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ADVANCES IN OLYMPIAD INEQUALITIES

Principles and Techniques for Old and New Problems

Preface

Inequalities have been extensively studied for at least a couple of centuries. Cauchy was among the first of the mathematicians who had major contributions in this literature. But it was probably not until Hardy et al.[1] we understood that it could be possible to study inequalities in a more systematic way. Since then a good number of books have discussed different aspects of inequalities for example, Beckenbach and Bellman.[2]

Objectives

- Dis

[1] G. H. Hardy et al.

1934 *Inequalities*, Cambridge University Press, ISBN: 0-521-35880-9.

[2] Edwin F. Beckenbach and Richard Bellman

1983 *Inequalities*, Springer.

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CHAPTER 1.

CLASSICAL INEQUALITIES

1.1 Introduction

Let us start with the most fundamental inequality.

$$x^2 \geqslant 0 \tag{1.1}$$

The first author calls it the mother of all inequality. Equality occurs if and only if $x = 0$. We can extend it for n variables.

$$x_1^2 + \ldots + x_n^2 \geqslant 0 \tag{1.2}$$

Equality occurs if $x_i = 0$ for all $1 \leqslant i \leqslant n$. We immediately get some useful results substituting x with appropriate expressions. Substituting x with $a - b$, we get

$$(a-b)^2 \geqslant 0$$
$$a^2 + b^2 \geqslant 2ab$$

This is true for any real numbers a, b and equality occurs if $a = b$. If a, b are positive, then replacing a and b by $\sqrt{a}$ and $\sqrt{b}$ respectively, we get

$$a + b \geqslant 2\sqrt{ab}$$
$$\iff \sqrt{\frac{a}{b}} + \sqrt{\frac{b}{a}} \geqslant 2$$
$$\iff x + \frac{1}{x} \geqslant 2$$

where $x = \frac{a}{b}$. This can be generalized to the following result.

Theorem 1 (*Arithmetic-Geometric Inequality*). *Let $a_1, \ldots, a_n$ be positive real numbers. Then*

$$\frac{a_1 + \ldots + a_n}{n} \geqslant \sqrt[n]{a_1 \cdots a_n}$$
$$\iff a_1 + \ldots + a_n \geqslant n\sqrt[n]{a_1 \cdots a_n}$$
$$\iff \left(\frac{a_1 + \ldots + a_n}{n}\right)^n \geqslant a_1 \cdots a_n$$

We will show a classical proof of this result here. The proof is due to Cauchy.[1] Later in Section 1.5, Section 2.3 we will show more proofs.

Proof. □

Note the following.

$$\frac{a+b}{2} \geqslant \sqrt{ab}$$

Now, $\frac{a+b}{2}$ is the *arithmetic mean* of a and b. On the right side, $\sqrt{ab}$ is the *geometric mean* of a and b. So the inequality states that the arithmetic mean of two positive real numbers is greater than or equal to their geometric mean. We can also rewrite it as the following.

$$\begin{aligned}\sqrt{ab}\left(\frac{a+b}{2}\right) &\geqslant ab\\ \sqrt{ab} &\geqslant \frac{2ab}{a+b}\\ \sqrt{ab} &\geqslant \frac{2}{\frac{1}{a}+\frac{1}{b}}\end{aligned}$$

$\dfrac{2ab}{a+b}$ is the *harmonic mean* of a and b. So, this form of the inequality states that the geometric mean is larger than the harmonic mean. This can be extended for three variables.

$$\frac{a+b+c}{3} \geqslant \frac{3}{\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}}$$

In fact, this result can be extensively generalized. First, we can consider n variables $a_1, \ldots, a_n$ instead of just a and b. Second, we can generalize the fact that $AM \geqslant GM \geqslant HM$. We will talk about this generalization in Section 1.5.

Problem 1.1. If a is a real number greater than one, prove that $\log a + \log_a e \geqslant 2$.

Throughout the book, if the base of logarithm is unspecified, then $\log a$ shall mean $\log_e a$. Also, try the next problem in a similar manner.

Problem 1.2. Prove the inequality

$$\frac{x^2}{1+x^4} \leqslant \frac{1}{2}$$

Recall from the definition of e that

$$\begin{aligned}e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\\ &= \sum_{i\geqslant 0} \frac{x^i}{i!}\end{aligned}$$

[1] Aug.-Louis Cauchy

1821 *Cours d'Analyse de l'école Royale Polytechnique*, L'Imprimerie Royale, Debure frères, Libraires du Roi et de la Bibliothèque du Roi.

You may also know from elementary differentiation that

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$$

In other words, if $x_n = \left(1+\frac{1}{n}\right)^n$, then x_n converges to e. However, we also have the relation that $x_n \leqslant x_{n+1}$. While we can prove this using induction, we will show a better proof with arithmetic-geometric mean inequality here.

Problem 1.3. Let $x_n = \left(1+\frac{1}{n}\right)^n$. Prove that $x_n \leqslant x_{n+1}$.

Solution. Apply the arithmetic-geometric mean inequality for $a_1 = 1, a_2 = 1+\frac{1}{n}, \ldots, a_{n+1} = 1+\frac{1}{n}$,

$$1+\left(1+\frac{1}{n}\right)+\ldots+\left(1+\frac{1}{n}\right) \geqslant (n+1)\sqrt[n+1]{1\cdot\left(1+\frac{1}{n}\right)\cdots\left(1+\frac{1}{n}\right)}$$

$$n+2 \geqslant (n+1)\sqrt[n+1]{\left(1+\frac{1}{n}\right)^n}$$

$$1+\frac{1}{n+1} \geqslant \sqrt[n+1]{\left(1+\frac{1}{n}\right)^n}$$

$$\left(1+\frac{1}{n+1}\right)^{n+1} \geqslant \left(1+\frac{1}{n}\right)^n$$

Note that we can generalize the idea used in this problem.

Problem 1.4. For positive real numbers x, y show that

$$\left(\frac{x+ny}{n+1}\right)^{n+1} \geqslant xy^n$$

Equality occurs only for $x = y$.

Similarly, we can show the following.

Problem 1.5. Let $y_n = \left(1+\frac{1}{n}\right)^{n+1}$. Show that $y_n \geqslant y_{n+1}$.

So y_n is decreasing. Using the two problems above, show that $2 < e < 4$. In fact, we can prove the following using elementary means

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} x_n = e$$

Problem 1.6. For a positive integer n, show that

$$\left(\frac{n+1}{2}\right)^n \geqslant n!$$

Equality occurs only for $n = 1$.

Solution. We use arithmetic-geometric inequality for $1, 2, \ldots, n$.

$$1+\ldots+n \geqslant n\sqrt[n]{1\cdots n}$$

$$\frac{n(n+1)}{2} \geqslant n\sqrt[n]{n!}$$

$$\frac{n+1}{2} \geqslant \sqrt[n]{n!}$$

Equality occurs if and only if $1 = \ldots = n$ which is possible only when $n = 1$.

Let us again go back to the mother of inequality. Setting $x_1 = a-b, x_2 = b-c, x_3 = c-a$ in 1.2,

$$(a-b)^2+(b-c)^2+(c-a)^2 \geqslant 0$$
$$\iff a^2+b^2+c^2-ab-bc-ca \geqslant 0$$
$$\iff a^2+b^2+c^2 \geqslant ab+bc+ca$$

We could prove this using $a^2+b^2 \geqslant 2ab$ repeatedly.

$$a^2+b^2 \geqslant 2ab$$
$$b^2+c^2 \geqslant 2bc$$
$$c^2+a^2 \geqslant 2ca$$
$$2(a^2+b^2+c^2) \geqslant 2(ab+bc+ca)$$
$$a^2+b^2+c^2-ab-bc-ca \geqslant 0$$

Multiplying both sides by $(a+b+c)$ and using the fact $a^3+b^3+c^3-3abc = (a+b+c)(a^2+b^2+c^2-ab-bc-ca)$,

$$a^3+b^3+c^3-3abc \geqslant 0$$
$$a^3+b^3+c^3 \geqslant 3abc$$

Replacing a^3, b^3, c^3 by u, v, w,

$$u+v+w \geqslant 3\sqrt[3]{uvw} \tag{1.3}$$

Before we go into any theory or method of solving problems, here are some basic tactics which are used very often for solving problems. Let $x, y, z > 0$ be real numbers.

(i) If $\frac{1}{x} \geqslant \frac{1}{y}$, then $\frac{1}{x-a} \geqslant \frac{1}{y}$ for $a \geqslant 0$. Similarly, $\frac{1}{x} \geqslant \frac{1}{y-b}$ for $b \geqslant 0$. They are true because $x-a \leqslant x$ implies $\frac{1}{x-a} \geqslant \frac{1}{x}$ and $y+b \geqslant y$ implies $\frac{1}{y+b} \leqslant \frac{1}{y}$.

(ii) Check if you can assume an ordering on the variables. For example, if the inequality is symmetric or cyclic on x, y, z, you may possibly assume without loss of generality that $x \geqslant y \geqslant z$ or $x \leqslant y \leqslant z$. We will discuss more on this in Section 2.2.

(iii) If you cannot assume an ordering on the variables e.g. $a \geqslant b \geqslant c$, can you assume that it has a *maximal element* e.g. $a = \max(a, b, c)$? Sometimes this helps in unexpected ways. See the example below.

(iv) See if you can get an *if and only if* way of proving an inequality. We will do this very often. If we can use if and only if (or *iff*), then we are free to prove either the if part or the only if part.

(v) Check if you can get some familiar expressions with some basic manipulations such as making the numerator or denominator equal. For example, see the transformation in 1.4. This often helps us get a clue on what to do with the inequality.

(vi) Does some substitutions such as $x = a+b, y = b+c, z = c+a$ or $x = a-b, y = b-c, z = c-a$ help? One could say we used substitution in our first proof of Nesbitt's inequality (see 1.5). Also, see the transformation $a - c = a - b + b - c$ used in the example below. We will check more on substitutions in Section 2.7.

(vii) Check if the inequality holds even if you put some restrictions on it. For example, you may be allowed to assume that one of the variables is 1 due to some scaling. We will talk about this in Section 2.6.

(viii) Can you reduce the number of variables without any assumption? See that we have already used it to prove $a^2 + b^2 + c^2 \geqslant ab + bc + ca$ in a proof above. Also, see a proof of Nesbitt's inequality below where it is enough to prove the inequality for reduced number of variables.

(ix) Induction works great in many cases. We will use induction on many occasions.

For demonstration purposes, let us prove $a^2 + b^2 + c^2 \geqslant ab + bc + ca$ again exploiting symmetry. Note that if we let

$$f(a, b, c) = a^2 + b^2 + c^2 - ab - bc - ca$$

then $f(a, b, c)$ is *symmetric* on a, b, c. We can verify this by the fact that $f(a, b, c) = f(b, a, c) = f(c, a, b)$ and so on. So, without loss of generality, we assume that $a \geqslant b \geqslant c$. Then see the following.

$$\begin{aligned}
a^2 + b^2 + c^2 &\geqslant ab + bc + ca\\
a(a-b) + b(b-c) - c(a-c) &\geqslant 0\\
a(a-b) + b(b-c) - c(a-b+b-c) &\geqslant 0\\
a(a-b) + b(b-c) - c(a-b) - c(b-c) &\geqslant 0\\
(a-c)(a-b) + (b-c)^2 &\geqslant 0
\end{aligned}$$

The last inequality immediately follows from the assumption that $a - c \geqslant 0, a - b \geqslant 0, (b-c)^2 \geqslant 0$. Also, note that we did not actually require the condition $a \geqslant b \geqslant c$. Just assuming $a = \max(a, b, c)$ was enough in this case to claim that the inequality holds.

THEOREM 2 (*Nesbitt's inequality*). *Let a, b, c be real positive numbers. Then*

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geqslant \frac{3}{2}$$

and equality occurs if and only $a = b = c$.

The thing about inequalities is that they can be solved in more than one ways most of the times. We will prove this inequality along with some other classical results such as *Cauchy-Schwarz inequality* in more than one ways. Some proofs will be discussed later when we develop some certain techniques. For now, we present some proofs using what we have already developed. First, we will try to *familiarize* the expression on the left side.

$$\begin{aligned}
S &= \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}\\
&= \frac{a+b+c}{b+c} - 1 + \frac{a+b+c}{c+a} - 1 + \frac{a+b+c}{a+b} - 1\\
&= (a+b+c)\left(\frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{a+b}\right) - 3\\
&= \frac{1}{2}(a+b+b+c+c+a)\left(\frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{a+b}\right) - 3 \qquad (1.4)
\end{aligned}$$

Classical proof. Setting

$$x=a+b, y=b+c, z=c+a \tag{1.5}$$

$$\begin{aligned}
S&=\frac{1}{2}(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)-3\\
&=\frac{1}{2}\left(1+\frac{x}{y}+\frac{x}{z}+\frac{y}{x}+1+\frac{y}{z}+\frac{z}{x}+\frac{z}{y}+1\right)-3\\
&=\frac{3}{2}+\frac{1}{2}\left(\frac{x}{y}+\frac{y}{x}+\frac{y}{z}+\frac{z}{y}+\frac{z}{x}+\frac{x}{z}\right)-3\\
&=\frac{3}{2}+\frac{1}{2}\left(u+\frac{1}{u}+v+\frac{1}{v}+w+\frac{1}{w}\right)-3
\end{aligned}$$

where $u=\frac{x}{y}, v=\frac{y}{z}, w=\frac{z}{x}$. Evidently, $u+\frac{1}{u}\geqslant 2, v+\frac{1}{v}\geqslant 2, w+\frac{1}{w}\geqslant 2$ and we have

$$S\geqslant\frac{3}{2}+\frac{1}{2}(2+2+2)-3$$

Thus, $S\geqslant\frac{3}{2}$. Equality occurs if $u=1, v=1, z=1$ or $x=y=z$ or $a=b=c$. □

Proof by arithmetic-harmonic mean inequality. We write the arithmetic-harmonic mean inequality for u, v, w as below.

$$\begin{aligned}
\frac{u+v+w}{3}&\geqslant\frac{3}{\frac{1}{u}+\frac{1}{v}+\frac{1}{v}}\\
\iff (u+v+w)\left(\frac{1}{u}+\frac{1}{v}+\frac{1}{w}\right)&\geqslant 9
\end{aligned}$$

Using this on 1.4,

$$S\geqslant\frac{1}{2}\cdot 9-2$$

□

Variable reduction proof. Let us clear the denominators in the original inequality.

$$\begin{aligned}
&\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geqslant\frac{3}{2}\\
\iff\ &2(a^3+b^3+c^3)\geqslant a^2b+ab^2+b^2c+bc^2+c^2a+ca^2
\end{aligned}$$

Notice the cyclic nature in the expression on the right side. There are 6 terms on the right side and if we count each of a^3, b^3, c^3 twice, there are 6 terms on the left side as well. So rearranging the inequality above as below

$$(a^3+b^3)+(b^3+c^3)+(c^3+a^3)\geqslant(a^2b+ab^2)+(b^2c+bc^2)+(c^2a+ca^2)$$

tells us that if we can prove $x^3+y^3\geqslant x^2y+xy^2$, we will be done if we simply sum them up for $(x,y)=(a,b),(b,c),(c,a)$. Here, we reduced the inequality from 3 variables to 2. And fortunately, this inequality is a lot easier to prove.

$$\begin{aligned}
x^3+y^3&\geqslant x^2y+xy^2\\
\iff x(x^2-y^2)-y(x^2-y^2)&\geqslant 0\\
\iff (x-y)(x^2-y^2)&\geqslant 0\\
\iff (x-y)^2(x+y)&\geqslant 0
\end{aligned}$$

The last inequality is evidently true. □

Proof using arithmetic-geometric mean inequality. Using 1.3,

$$(a+b)+(b+c)+(c+a) \geqslant 3\sqrt[3]{(a+b)(b+c)(c+a)}$$
$$\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a} \geqslant 3\sqrt[3]{\frac{1}{(a+b)(b+c)(c+a)}}$$

We can use this observation on 1.4 and get the following.

$$S \geqslant \frac{1}{2}\cdot 3\sqrt[3]{(a+b)(b+c)(c+a)}\cdot 3\sqrt[3]{\frac{1}{(a+b)(b+c)(c+a)}}-2$$
$$\geqslant \frac{9}{2}-2=\frac{3}{2}$$

This again proves the inequality. □

ANOVIĆ and PEČARIĆ[2] proves the following generalization of *Nesbitt's inequality*.

THEOREM 3 (Generalization of Nesbitt's inequality). *Let x, y, z be positive real numbers and*

$$M = \max\{\frac{x}{y+z}+\frac{2(y+z)}{2x+y+z}, \frac{y}{z+x}+\frac{2(z+x)}{2y+z+x}, \frac{z}{x+y}+\frac{2(x+y)}{2z+x+y}\}$$
$$m = \min\{\frac{x}{y+z}+\frac{2(y+z)}{2x+y+z}, \frac{y}{z+x}+\frac{2(z+x)}{2y+z+x}, \frac{z}{x+y}+\frac{2(x+y)}{2z+x+y}\}$$

Then

$$\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y} \geqslant M \geqslant m \geqslant \frac{3}{2}$$

We have already showed that $a+b \geqslant 2\sqrt{ab}$ which is a special case of the arithmetic-geometric mean inequality. Let us consider the following question. We are given four positive real numbers a, b, c, d such that $S = a+b = c+d$. Which of the products between ab and cd is the smaller one?

$$\begin{aligned} 4ab &= (a+b)^2-(a-b)^2 \\ &= S^2-(a-b)^2 \\ 4cd &= (c+d)^2-(c-d)^2 \\ &= S^2-(c-d)^2 \end{aligned}$$

As we can see here, the sign in $4ab?4cd$ (? to be replaced by one of $>, <, \geqslant, \leqslant$) will be dictated by which of the differences $a-b, c-d$ is smaller. Since $x^2 \geqslant 0$, if $(a-b)^2 < (c-d)^2$, we have $4ab > 4cd$. Now, consider the product $a_1 \cdots a_n$. We want to see how the product changes as a_i varies with respect to $\bar{a} = \frac{a_1+\ldots+a_n}{n}$.

If all the a_i are equal to each other, then we have nothing to check. Otherwise, there are at least two positive integers i and j such that a_i and a_j are not equal to $\bar{a}$. Moreover, one of them is greater than $\bar{a}$ and the other is smaller than $\bar{a}$ because all of them cannot be greater (or smaller) than $\bar{a}$. Without loss

[2] S. IVELIĆ BRA ANOVIĆ and JOSIP PEČARIĆ
2011 "Generalizations of converse Jensen's inequality and related results", *Journal of Mathematical Inequalities*, no. 1, pp. 43-60, DOI: 10.7153/jmi-05-06.

of generality, assume that a_1, a_2 are those two numbers and $a_1 = \bar{a} - h, a_2 = \bar{a} + k$. Now, consider two other positive numbers c and d which keeps the sum fixed, for example $c = \bar{a}, d = \bar{a} + k - h$. We have

$$\begin{aligned} cd &= \bar{a}(\bar{a} + k - h) \\ &= \bar{a}^2 + \bar{a}k - \bar{a}h \\ a_1a_2 &= (\bar{a} - h)(\bar{a} + k) \\ &= \bar{a}^2 + \bar{a}k - \bar{a}h - hk \\ &= cd - hk < cd \\ a_1a_2\cdots a_n &< cd\cdots a_n \end{aligned}$$

This basically tells us that we can increase the product further if there are other a_i which are not equal to $\bar{a}$ and the product is maximum when all a_i is equal to $\bar{a}$.

Problem 1.7. Show that for a positive integer n,

$$n! > \left(\frac{n}{4}\right)^n$$

We can easily prove this with induction. This can be improved to $n! > \left(\frac{n}{3}\right)^n$. In fact, we can prove the following.

$$n! > \left(\frac{n}{e}\right)^n$$

This is another nice result. For example, setting $n = 2019$,

$$2019! > 673^{2019}$$

We can even bound $n!$ from both sides with the next result.

Problem 1.8. For a positive integer n, prove the inequality

$$e\left(\frac{n+1}{e}\right)^{n+1} > n! > \left(\frac{n}{e}\right)^n$$

Equality is not possible because on both sides we have non-integers.

Problem 1.9. Let x, y, z be positive real numbers. Prove that

$$\frac{2}{x+y} + \frac{2}{y+z} + \frac{2}{z+x} \geqslant \frac{9}{x+y+z}$$

Solution.

A function f is *non-decreasing* (resp. *increasing*) on the interval I if for any $a, b \in I$, $(a-b)(f(a) - f(b)) \geqslant 0$ (resp. $(a-b)(f(a)-f(b)) > 0$). Similarly, f is *non-increasing* (resp. *decreasing*) on the interval I if for any $a, b \in I$, $(a-b)(f(a)-f(b)) \leqslant 0$ (resp. $(a-b)(f(a)-f(b)) < 0$). If either of these two conditions apply for f, then f is a *monotone function*. We say that f is *monotonic*. Note that the slope between any two points $(a, f(a))$ and $(b, f(b))$ is $m = \frac{f(a)-f(b)}{a-b}$ and the quantity we have used for the definition is

$$\begin{aligned} (a-b)(f(a)-f(b)) &= (a-b)^2\frac{f(a)-f(b)}{f(a)-f(b)} \\ &= (a-b)^2 m \end{aligned}$$

So, the sign of this quantity is the same as the sign of the slope m.

A sequence (a_n) is *non-decreasing* if $a_i \leqslant a_{i+1}$ for all $i \in \mathbb{N}$. (a_n) is *strictly increasing* if $a_i < a_{i+1}$. Similarly, (a_n) is *non-increasing* if $a_i \geqslant a_{i+1}$ for all i. (a_n) is *strictly decreasing* if $a_i > a_{i+1}$. If (a_n) is either increasing or decreasing, then (a_n) is a *monotone sequence*. We say that (a_n) is *monotonic*.

Theorem 4 *(Abel formula). Let (a_n) and (b_n) be two sequences of real numbers. If $c_k = b_1 + ... + b_k$ for $1 \leqslant k \leqslant n$, then*

$$a_1b_1 + ... + a_nb_n = (a_1 - a_2)c_1 + (a_2 - a_3)c_2 + ... + (a_{n-1} - a_n)c_n + a_nc_n$$

Proof. □

Theorem 5 *(Abel's Inequality). Let (a_n) and (b_n) be two sequences of real numbers such that $b_1 \geqslant ... \geqslant b_n \geqslant 0$. For $1 \leqslant k \leqslant n$, define*

$$\begin{aligned} s_k &= a_1 + ... + a_k \\ m &= \min_{1 \leqslant i \leqslant n} (s_i) \\ M &= \max_{1 \leqslant i \leqslant n} (s_i) \end{aligned}$$

Then we have

$$mb_1 \leqslant a_1b_1 + ... + a_nb_n \leqslant Mb_1$$

Proof. Write the sum $a_1b_1 + ... + a_nb_n$ as the following.

$$\begin{aligned} a_1b_1 + ... + a_nb_n &= s_1b_1 + (s_2 - s_1)b_2 + ... + (s_n - s_{n-1})b_n \\ &= s_1(b_1 - b_2) + s_2(b_2 - b_3) + ... + s_{n-1}(b_{n-1} - b_n) + s_nb_n \end{aligned}$$

Using $m \leqslant s_i \leqslant M$,

$$\begin{gathered} m(b_1 - b_2) \leqslant s_1(b_1 - b_2) \leqslant M(b_1 - b_2) \\ \vdots \\ m(b_{n-1} - b_n) \leqslant s_{n-1}(b_{n-1} - b_n) \leqslant M(b_{n-1} - b_n) \end{gathered}$$

We additionally have $mb_n \leqslant s_nb_n \leqslant Mb_n$. Summing these inequalities together,

$$\begin{aligned} m(b_1 - b_2 + b_2 - b_3 + ... + b_{n-1} - b_n) + mb_n &\leqslant s_1(b_1 - b_2) \\ &\quad + ... + s_{n-1}(b_{n-1} - b_n) + s_nb_n \\ &\leqslant M(b_1 - b_2 + b_2 - b_3 + ... \\ &\quad + b_{n-1} - b_n) + Mb_n \\ mb_1 &\leqslant a_1b_1 + ... + a_nb_n \leqslant Mb_1 \end{aligned}$$

This proves the inequality. □

1.2 Warm Up Problems

Try the following problems as warm up exercises.

Problem 1.10. For real numbers x, y, prove that

$$|x+y| \leqslant |x|+|y|$$
$$||x|-|y|| \leqslant |x+y|$$
$$(|x|-|y|)^2 \leqslant |x^2-y^2|$$

When does equality occur in the last inequality?

The first inequality in this problem is also known as the *triangle inequality*.

Problem 1.11. Prove *Jordan's inequality*.

$$\frac{2}{\pi} \leqslant \frac{\sin\theta}{\theta} < 1$$

for $0 < |\theta| < \frac{\pi}{2}$.

Problem 1.12. Prove *Redheffer's inequality*.

$$\frac{\sin\theta}{\theta} \geqslant \frac{\pi^2-\theta^2}{\pi^2+\theta^2}$$

The two inequalities above are not derived from each other. Also, you may have to put a bit of extra effort to prove them. Use calculus if that gives a faster solution.

Problem 1.13 (*IMO* 1960). For which real value of x does the inequality

$$\frac{4x^2}{(1-\sqrt{1+2x})^2} < 2x+9$$

hold?

1.3 Cauchy-Schwarz Inequality and Improvements

Cauchy-Schwarz inequality also known as *Cauchy-Bunyakovsky-Schwarz* inequality is among the most important results for solving problems.

Theorem 6 (*Cauchy-Bunyakovsky-Schwarz inequality*). *Let n be a positive integer, $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ be real numbers. Then*

$$(a_1^2+\ldots+a_n^2)(b_1^2+\ldots+b_n^2) \geqslant (a_1b_1+\ldots+a_nb_n)^2 \tag{1.6}$$
$$\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) \geqslant \left(\sum_{i=1}^{n} a_ib_i\right)^2$$

and equality holds if and only if $\frac{a_1}{b_1} = \cdots = \frac{a_n}{b_n}$.

The reason why it is also called Cauchy-Bunyakovsky-Schwarz inequality is that the analog of this inequality for integrals

$$\left(\int_a^b f(x)g(x)dx\right)^2 \leqslant \left(\int_a^b f^2(x)dx\right) \cdot \left(\int_a^b g^2(x)dx\right) \tag{1.7}$$

appeared in the *Mémoire* BUNYAKOVSKY[3] for the first time. This *Mémoire* was published by the Imperial Academy of Sciences of St. Petersburg in BUNYAKOVSKY.[4] STEELE[5] states that *Bunyakovsky (1804–1889) had studied in Paris with Cauchy, and he was quite familiar with Cauchy's work on inequalities; so much so that by the time he came to write his Mémoire, Bunyakovsky was content to refer to the classical form of Cauchy' inequality for finite sums simply as well-known.*

This theorem has many proofs and even more applications. We will see later in Section 2.1 how powerful just a special case of this inequality can be. Let us start with a proof that is probably the most elegant one. Before we show proofs, we will introduce a notation for shortening the *Cauchy-Bunyakovsky-Schwarz inequality*. Let two vectors (think of a vector as an ordered list of numbers) be $\mathrm{a} = (a_1, \dots, a_n)$ and $\mathrm{b} = (b_1, \dots, b_n)$. The numbers $a_1, \dots, a_n$ inside the vector a can be called *elements* of a. So, a has n elements and so does b. Denote by $\langle \mathrm{a}, \mathrm{b} \rangle$ the *inner product* of a and b defined as

$$\langle \mathrm{a}, \mathrm{b} \rangle = a_1 b_1 + \dots + a_n b_n$$

Then the theorem can be stated as

$$\langle \mathrm{a}, \mathrm{b} \rangle^2 \leqslant \langle \mathrm{a}, \mathrm{a} \rangle \cdot \langle \mathrm{b}, \mathrm{b} \rangle$$

The inner product has some interesting properties. Note that the mother of inequality in Equation 1.2 can be stated as

$$\langle \mathrm{x}, \mathrm{x} \rangle \geqslant 0$$

for any $\mathrm{x} = (x_1, \dots, x_n)$ where x_i is a real number. Similarly, define $\mathrm{y} = (y_1, \dots, y_n)$. Then we have the following.

(i) $\langle \mathrm{x}, \mathrm{x} \rangle = 0$ if and only if $\mathrm{x} = 0$ i.e. $\mathrm{x} = (0, \dots, 0)$.

(ii) For any real number α, $\langle \alpha\mathrm{x}, \mathrm{y} \rangle = \alpha\langle \mathrm{x}, \mathrm{y} \rangle$.

(iii) $\langle \mathrm{x}, \mathrm{y} \rangle = \langle \mathrm{y}, \mathrm{x} \rangle$.

(iv) For $\mathrm{z} = (z_1, \dots, z_n)$, $\langle \mathrm{x}, \mathrm{y} + \mathrm{z} \rangle = \langle \mathrm{x}, \mathrm{y} \rangle + \langle \mathrm{x}, \mathrm{z} \rangle$.

(v) $\mathrm{x} + \mathrm{y} = (x_1 + y_1, \dots, x_n + y_n)$.

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[3] VIKTOR YAKOVLEVICH BUNYAKOVSKY

1846 *Foundations of the mathematical theory of probabilities*, Imperial Academy of Sciences of St. Petersburg, p. 495, DOI: http://books.e-heritage.ru/book/10070419, Page 4.

[4] VIKTOR YAKOVLEVICH BUNYAKOVSKY

1859 "Sur quelques inégalités concernant les intégrales ordinaires et les intégrales aux différences finies", *Mémoires de l'Acad. de St.-Pétersbourg*, 7th ser., vol. I, no. 9, pp. 1-18.

[5] JOHN MICHAEL STEELE

2010 *The Cauchy-Schwarz master class: an introduction to the art of mathematical inequalities*, Cambridge University Press, Page 10.

For convenience, we can also define x^r as

$$\mathrm{x}^r = (x_1^r, \dots, x_n^r)$$

for a real number r. Let us generalize this notation further. Let m be a positive integer. Then for $\mathrm{x} = (x_1, \dots, x_n)$, define

$$\begin{aligned} \|\mathrm{x}\|_m &= \langle \underbrace{\mathrm{x}, \dots, \mathrm{x}}_{m \text{ times}} \rangle^{\frac{1}{m}} \\ &= \sqrt[m]{x_1^m + \dots + x_n^m} \end{aligned}$$

For $m = 2$, we will simply write $\|\mathrm{x}\|$ instead of $\|\mathrm{x}\|_2$ just like we write $\sqrt{x}$ instead of $\sqrt[2]{x}$. So we can write *Cauchy-Bunyakovsky-Schwarz inequality* in a more compact form.

$$\|\mathrm{a}\| \cdot \|\mathrm{b}\| \geqslant \langle \mathrm{a}, \mathrm{b} \rangle$$

This notion can be generalized even further for arbitrary real numbers. For a real number p, define

$$\|\mathrm{x}\|_p = \sqrt[p]{x_1^p + \dots + x_n^p}$$

This is often called the L_p *norm* for the vector x. Furthermore, we can define the addition or subtraction between two vectors as

$$\begin{aligned} \mathrm{a} \pm \mathrm{b} &= (a_1 \pm b_1, \dots, a_n \pm b_n) \\ m\mathrm{a} &= (ma_1, \dots, ma_n) \end{aligned}$$

These notations will help us shorten some long inequalities later. We will see the benefit of these notations shortly in *Hölder's inequality* and *Minkowski's Inequality*. Let us see the proofs now.

Proof by vector. Consider two vectors $\mathrm{a} = (a_1, \dots, a_n), \mathrm{b} = (b_1, \dots, b_n)$ and their modulus

$$\|\mathrm{a}\| = \sqrt{a_1^2 + \dots + a_n^2}$$

$$\|\mathrm{b}\| = \sqrt{b_1^2 + \dots + b_n^2}$$

If θ is the minimum angle between them, from the rule of *dot product* we get

$$\begin{aligned} \mathrm{a} \cdot \mathrm{b} &= a_1 b_1 + \dots + a_n b_n \\ \|\mathrm{a}\| \cdot \|\mathrm{b}\| \cdot \cos\theta &= a_1 b_1 + \dots + a_n b_n \\ \cos\theta &= \frac{a_1 b_1 + \dots + a_n b_n}{\sqrt{a_1^2 + \dots + a_n^2}\sqrt{b_1^2 + \dots + b_n^2}} \end{aligned}$$

Since $-1 \leqslant \cos\theta \leqslant 1$, after squaring we get

$$\begin{aligned} \cos^2\theta &\leqslant 1 \\ \left(\frac{a_1 b_1 + \dots + a_n b_n}{\sqrt{a_1^2 + \dots + a_n^2}\sqrt{b_1^2 + \dots + b_n^2}}\right)^2 &\leqslant 1 \\ \iff (a_1^2 + \dots + a_n^2)(b_1^2 + \dots + b_n^2) &\geqslant (a_1 b_1 + \dots + a_n b_n)^2 \end{aligned}$$

Equality occurs if and only $\cos\theta = 1$ or when a and b are parallel. In other words, when we have

$$\frac{a_1}{b_1} = \dots = \frac{a_n}{b_n}$$

□

Classical proof by Schwarz. Consider the quadratic polynomial

$$\begin{aligned} P(x) &= \sum_{i=1}^{n}(a_ix-b_i)^2 \\ &= x^2\sum_{i=1}^{n}a_i^2 - 2x\sum_{i=1}^{n}a_ib_i + \sum_{i=1}^{n}b_i^2 \\ &= Ax^2 - Bx + C \end{aligned}$$

where $A = \sum_{i=1}^{n} a_i^2, B = 2\sum_{i=1}^{n} a_ib_i, C = \sum_{i=1}^{n} b_i^2$. Setting $x_i \to a_ix - b_i$ in 1.2, we see that $P(x) \geqslant 0$. Then the discriminant of P must be $\leqslant 0$.

$$\begin{aligned} B^2 - 4AC &\leqslant 0 \\ 4(a_1b_1 + \ldots + a_nb_n)^2 - 4(a_1^2 + \ldots + a_n^2)(b_1^2 + \ldots + b_n^2) &\leqslant 0 \\ \iff (a_1^2 + \ldots + a_n^2)(b_1^2 + \ldots + b_n^2) &\geqslant (a_1b_1 + \ldots + a_nb_n)^2 \end{aligned}$$

Equality occurs if and only if $P(x) = 0$ for some x. Then

$$x = \frac{b_1}{a_1} = \cdots = \frac{b_n}{a_n}$$

□

Proof by arithmetic-geometric mean. We use the notations from the proof above. If $A = 0$ or $B = 0$, then the inequality is an identity. So, assume that $A, B \neq 0$. Note the following.

$$\begin{gathered} \frac{a_1^2}{A} + \frac{b_1^2}{B} \geqslant 2\frac{a_1b_1}{\sqrt{AB}} \\ \vdots \\ \frac{a_n^2}{A} + \frac{b_n^2}{B} \geqslant 2\frac{a_nb_n}{\sqrt{AB}} \end{gathered}$$

Summing them together,

$$\begin{aligned} \frac{a_1^2}{A} + \frac{b_1^2}{B} + \ldots + \frac{a_n^2}{A} + \frac{b_1^2}{B} &\geqslant 2\frac{a_1b_1 + \ldots + a_nb_n}{\sqrt{AB}} \\ \frac{a_1^2 + \ldots + a_n^2}{A} + \frac{b_1^2 + \ldots + b_n^2}{B} &\geqslant 2\frac{a_1b_1 + \ldots + a_nb_n}{\sqrt{AB}} \\ 1 + 1 &\geqslant 2\frac{a_1b_1 + \ldots + a_nb_n}{\sqrt{AB}} \\ AB &\geqslant \left(a_1b_1 + \ldots + a_nb_n\right)^2 \end{aligned}$$

Again, equality holds if and only if $\frac{a_i}{A} = \frac{b_i}{B}$ or $\frac{a_i}{b_i} = \frac{A}{B} = c$, a constant for all i. □

Proof using the mother of inequality. Let $x_i = \frac{a_i}{A}, y_i = \frac{b_i}{B}$ for $1 \leqslant i \leqslant n$. We have

$$\sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} y_i^2 = 1$$

Rewrite the inequality as

$$\begin{aligned} x_1y_1 + \dots + x_ny_n &\leqslant 1 \\ \iff 2(x_1y_1 + \dots + x_ny_n) &\leqslant 2 \\ \iff 2(x_1y_1 + \dots + x_ny_n) &\leqslant x_1^2 + \dots + x_n^2 + y_1^2 + \dots + y_n^2 \\ \iff (x_i - y_i)^2 + \dots + (x_n - y_n)^2 &\geqslant 0 \end{aligned}$$

This is obviously true. □

Proof using sequences. Define the sequence (S_n) as

$$\begin{aligned} D_n &= a_1b_1 + \dots + a_nb_n \\ A_n &= a_1^2 + \dots + a_n^2 \\ B_n &= b_1^2 + \dots + b_n^2 \\ S_n &= D_n^2 - A_nB_n \end{aligned}$$

We want to show that $S_n \leqslant 0$. See the following.

$$\begin{aligned} D_{n+1}^2 - D_n^2 &= (D_n + a_{n+1}b_{n+1})^2 - D_n^2 \\ &= 2a_{n+1}b_{n+1}D_n + a_{n+1}^2b_{n+1}^2 \\ A_{n+1}B_{n+1} - A_nB_n &= (A_n + a_{n+1}^2)(B_n + b_{n+1}^2) - A_nB_n \\ &= a_{n+1}^2B_n + b_{n+1}^2A_n + a_{n+1}^2b_{n+1}^2 \\ S_{n+1} - S_n &= D_{n+1}^2 - A_{n+1}B_{n+1} - D_n^2 + A_nB_n \\ &= 2a_{n+1}b_{n+1}D_n + a_{n+1}^2b_{n+1}^2 \\ &- a_{n+1}^2B_n - b_{n+1}^2A_n - a_{n+1}^2b_{n+1}^2 \\ &= 2a_{n+1}b_{n+1}D_n - a_{n+1}^2B_n - b_{n+1}^2A_n \\ &= 2a_{n+1}b_{n+1}(D_{n-1} + a_nb_n) - a_{n+1}^2(B_{n-1} \\ &+ b_{n+1}^2) - b_{n+1}^2(A_{n-1} + a_{n+1}^2) \\ &= 2a_{n+1}b_{n+1}D_{n-1} + 2a_{n+1}b_{n+1}a_nb_n - a_{n+1}^2B_{n-1} \\ &- a_{n+1}^2b_n^2 - b_{n+1}^2A_{n-1} - a_{n+1}^2b_n^2 \\ &= 2a_{n+1}b_{n+1}D_{n-1} - a_{n+1}^2B_{n-1} \\ &- b_{n+1}^2A_{n-1} - (a_{n+1}b_n - a_nb_{n+1})^2 \\ &\vdots \\ &= -((a_{n+1}b_n - a_nb_{n+1})^2 + \dots + (a_{n+1}b_1 - a_1b_{n+1})^2) \end{aligned}$$

Clearly, $S_{n+1} - S_n \leqslant 0$, so

$$S_{n+1} \leqslant S_n \leqslant \cdots S_1 = 0$$

This proves the inequality. □

The fact about $S_{n+1} - S_n$ also proves the *Lagrange Identity*.

$$\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) - \left(\sum_{i=1}^{n} a_ib_i\right)^2 = \sum_{1 \leqslant i < j \leqslant n} (a_ib_j - a_jb_i)^2$$

This equation immediately proves the theorem but it is not quite obvious how the identity follows unless someone knows it beforehand. For this reason, some people also call this the Lagrange inequality but we will stick with Cauchy-Schwarz.

Gram's inequality is a very nice generalization of *Cauchy-Bunyakovsky-Schwarz inequality*.

THEOREM 7 *(Gram's inequality). Let* $\mathrm{x}_1 = (x_{11}, \dots, x_{1n}), \dots, \mathrm{x}_n = (x_{n1}, \dots, x_{nn})$ *be vectors with n elements. Then*

$$\begin{vmatrix} \langle \mathrm{x}_1, \mathrm{x}_1 \rangle & \dots & \langle \mathrm{x}_1, \mathrm{x}_n \rangle \\ & \dots & \\ \langle \mathrm{x}_n, \mathrm{x}_1 \rangle & \dots & \langle \mathrm{x}_n, \mathrm{x}_n \rangle \end{vmatrix} \geqslant 0$$

Equality occurs if and only if there are real numbers $a_1, \dots, a_n$ *such that all of them are not zero at the same time and*

$$a_1\mathrm{x}_1 + \dots + a_n\mathrm{x}_n = 0$$

where 0 *is the zero vector with every element* 0.

This result is very intuitive to say the least. Note that we get *Cauchy-Bunyakovsky-Schwarz inequality* for $n = 2$ since

$$\begin{vmatrix} \langle \mathrm{x}, \mathrm{x} \rangle & \langle \mathrm{x}, \mathrm{y} \rangle \\ \langle \mathrm{y}, \mathrm{x} \rangle & \langle \mathrm{y}, \mathrm{y} \rangle \end{vmatrix} \geqslant 0$$

$$\iff \langle \mathrm{x}, \mathrm{x} \rangle \cdot \langle \mathrm{y}, \mathrm{y} \rangle - \langle \mathrm{x}, \mathrm{y} \rangle \cdot \langle \mathrm{y}, \mathrm{x} \rangle \geqslant 0$$

implies *Cauchy-Bunyakovsky-Schwarz inequality* due to the fact $\langle \mathrm{x}, \mathrm{y} \rangle = \langle \mathrm{y}, \mathrm{x} \rangle$. The condition of equality in Gram's inequality is *linear dependence*. Let $\mathrm{x}_1, \dots, \mathrm{x}_n$ be some vectors with n elements. Then they are *linearly dependent* if and only if there are real numbers $a_1, \dots, a_n$ not all zero such that

$$a_1\mathrm{x}_1 + \dots + a_n\mathrm{x}_n = 0$$

If such $a_1, \dots, a_n$ do not exist, that is the condition is satisfied only when $a_1 = \dots = a_n = 0$, then $\mathrm{x}_1, \dots, \mathrm{x}_n$ are *linearly independent*.

1.4 Complex Numbers

We may occasionally need complex numbers. So we provide a brief introduction here. A complex number z is defined as

$$z = x + iy$$

where x, y are real numbers and i is the imaginary unit such that $i^2 = -1$. We call x the real component and y the complex component of z. Complex numbers follow the same properties as vectors which we established in Section 1.3. That is for two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$,

$$\begin{aligned} z_1 \pm z_2 &= (x_1 \pm x_2) + i(y_1 \pm y_2) \\ \langle z_1, z_2 \rangle &= x_1x_2 + y_1y_2 \end{aligned}$$

Here $\langle z_1, z_2 \rangle$ is the dot product of z_1 and z_2. We define an additional operation $\cdot$ as

$$z_1 \cdot z_2 = x_1 y_2 - x_2 y_1$$

The *modulus* of z is similar to L_2 norm.

$$|z| = \sqrt{x^2 + y^2}$$

We have the following property.

$$|z_1 z_2| = |z_1| \cdot |z_2|$$

For a complex number $z = x + iy$, the *conjugate* of z is defined as

$$\bar{z} = x - iy$$

So, we have $z\bar{z} = |z|^2 = |\bar{z}|^2$. The *argument* of z is defined as

$$\arg(z) = \tan\left(\frac{y}{x}\right)$$

You can think of it as the angle the point (x, y) creates with the positive X axis and the origin.

Problem 1.14. Prove the triangle inequality for complex numbers.

$$|z_1 + z_2| \leqslant |z_1| + |z_2|$$

This inequality can be generalized as the following.

$$|z_1 + \ldots + z_n| \leqslant |z_1| + \ldots + |z_n|$$

When does equality occur?

1.5 Bernoulli and Power Mean Inequality

The *Bernoulli inequality* is a well known result and it has some nice consequences. We mentioned earlier that we will show more proofs of arithmetic-geometric mean inequality. We will show one such proof here using this inequality.

Theorem 8 (*Bernoulli's inequality*). *Let n be a positive integer and $x > -1$ be a real number. Then*

$$(1 + x)^n \geqslant 1 + nx$$

This inequality can be generalized as the following result.

Theorem 9 (*Generalized Bernoulli inequality*). *Let $x_1, \ldots, x_n > -1$ be real numbers such that either all are positive or all are negative. Then*

$$(1 + x_1) \cdots (1 + x_n) > 1 + x_1 + \ldots + x_n$$

Let $a_1, \dots, a_n$ be positive real numbers. Then the *generalized mean* or *power mean of order r* is defined as

$$\mathfrak{M}_r(a_1, \dots, a_n) = \left(\frac{a_1^r + \dots + a_n^r}{n}\right)^{\frac{1}{r}}$$

Note that the arithmetic mean of $a_1, \dots, a_n$ is actually $\mathfrak{M}_1(a_1, \dots, a_n)$. Similarly, the harmonic mean is $\mathfrak{M}_{-1}(a_1, \dots, a_n)$. Moreover, $\mathfrak{M}_0(a_1, \dots, a_n)$ is the geometric mean which we show below.

We may omit the numbers $a_1, \dots, a_n$ and just call it $\mathfrak{M}_r$ instead of $\mathfrak{M}_r(a_1, \dots, a_n)$ if the context is clear. An even better way to denote this would be using $\mathfrak{M}_r(\mathrm{a})$ where $\mathrm{a} = (a_1, \dots, a_n)$.

$$\mathfrak{M}_r(\mathrm{a}) = \frac{\|\mathrm{a}\|_r}{\sqrt[r]{n}}$$

We can also denote the arithmetic, geometric and harmonic means of a by $\mathfrak{A}(\mathrm{a}), \mathfrak{G}(\mathrm{a})$ and $\mathfrak{H}(\mathrm{a})$ respectively. The notations $\mathfrak{M}, \mathfrak{A}, \mathfrak{G}, \mathfrak{H}$ are inspired by Hardy et al.[6]

Next, we show that $\mathfrak{G}(\mathrm{a})$ is the geometric mean.

$$\mathfrak{G}(\mathrm{a}) = \lim_{r \to 0} \left(\frac{a_1^r + \dots + a_n^r}{n}\right)^{\frac{1}{r}}$$

Then we can write arithmetic-geometric-harmonic mean inequality as $\mathfrak{A}(\mathrm{a}) \geqslant \mathfrak{G}(\mathrm{a}) \geqslant \mathfrak{H}(\mathrm{a})$. This is generalized in the next result.

Theorem 10 (*Power mean inequality*). *If r, s are real numbers such that $r \leqslant s$, then for a vector* a,

$$\min\{\mathrm{a}\} \leqslant \mathfrak{M}_r(\mathrm{a}) \leqslant \mathfrak{M}_s(\mathrm{a}) \leqslant \max\{\mathrm{a}\}$$

Equality occurs if and only if $r = s$.

This inequality can be extended further with the notion of what we call *weighted means*. Let $\omega = (w_1, \dots, w_n)$ be a vector of non-negative real numbers such that $w_1 + \dots + w_n = 1$. Then the weighted arithmetic mean of the real numbers $a_1, \dots, a_n$ is

$$\mathfrak{A}(\mathrm{a}, \omega) = w_1 a_1 + \dots + w_n a_n$$

In general, the *weighted power mean of order r* is

$$\mathfrak{M}_r(\mathrm{a}, \omega) = (w_1 a_1^r + \dots + w_n a_n^r)^{\frac{1}{r}}$$

If the context is clear on what the weights are, then we may omit the weight from the notation and simply write $\mathfrak{M}_r(\mathrm{a})$. Let us call ω a *weight vector* if $w_1, \dots, w_n \geqslant 0$ and $w_1 + \dots + w_n = 1$. We can convert almost any vector of non-negative real numbers into a weight vector. If $\tau = (t_1, \dots, t_n)$ is an arbitrary vector not all elements zero, then

$$\omega = \left(\frac{t_1}{t_1 + \dots + t_n}, \dots, \frac{t_n}{t_1 + \dots + t_n}\right)$$

is a weight vector. The power mean inequality applies to weighted means as well.

Theorem 11 (*Weighted Power Mean Inequality*). *Let* a *and ω be vectors with n elements. Then for real numbers r, s such that $r \leqslant s$,*

$$\min\{\mathrm{a}\} \leqslant \mathfrak{M}_r(\mathrm{a}, \omega) \leqslant \mathfrak{M}_s(\mathrm{a}, \omega) \leqslant \max\{\mathrm{a}\}$$

A special case of this is the *weighted arithmetic-geometric mean inequality.*

$$w_1 a_1 + \dots + w_n a_n \geqslant a_1^{w_1} \cdots a_n^{w_n} \tag{1.8}$$

[6] Hardy et al., *Inequalities* cit.

1.6 Hölder, Minkowski's Inequality

Theorem 12 (*Generalized Hölder's Inequality*). *Let* $x_1, \ldots, x_n$ *be vectors of n positive real numbers and* ω *be a weight vector. Then*

$$\mathfrak{G}\Big(\sum_{i=1}^{n} x_i, \omega\Big) \geqslant \sum_{i=1}^{n} \mathfrak{G}(x_i, \omega)$$

Equality occurs if any two of the vectors are proportional.

This is not exactly the Hölder's inequality to be precise. This inequality can be specified as Generalized Hölder's inequality which appears in Minkowski.[7] And Hölder[8] proved *Hölder's inequality* for which we need to define conjugates. Two real numbers u and v are *conjugate* if

$$\frac{1}{u} + \frac{1}{v} = 1$$

Theorem 13 (*Hölder's inequality*). *Let* $u > 1$ *be a real number and* v *be the conjugate of* u. *If* α, β *are complex numbers,*

$$\langle \alpha, \beta \rangle \leqslant \|\alpha\|_u \cdot \|\beta\|_v$$

Equality occurs if α^u *and* β^v *are proportional.*

Hölder's inequality can be proved with the help of Young.[9]

Theorem 14 (*Young's inequality*). *Let* a, b *be positive real numbers and* u, v *be conjugates. Then*

$$ab \leqslant \frac{a^u}{u} + \frac{b^v}{v}$$

Razminia[10] presents another proof of *Generalized Hölder's Inequality*. The next result is usually known as *the converse of Hölder's inequality*.

Theorem 15 (*The converse of Hölder's inequality*). *Let* a *and* b *be vectors of positive real numbers,* u *and* v *be conjugates. If* B *is a positive real number, then a necessary and sufficient condition that*

$$\sum_{i=1}^{n} a_i^u \leqslant A$$

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[7] Hermann Minkowski

1968 *Geometrie der Zahlen*, Johnson Reprint Corporation, Page 117.

[8] O. Hölder

1889 "Ueber einen Mittelwertsatz", German, *Gött. Nachr.*, vol. MDCCCLXXXIX, pp. 38-47.

[9] William Henry Young

1912 "On classes of summable functions and their Fourier Series", *Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character*, vol. LXXXVII, no. 594, pp. 225-229, DOI: 10.1098/rspa.1912.0076.

[10] K. Razminia

2019 "Hölder's inequality revisited", *The Mathematical Gazette*, vol. CIII, no. 558, pp. 512-514, DOI: 10.1017/mag.2019.117.

is that

$$\langle \mathrm{a}, \mathrm{b}\rangle \leqslant \sqrt[u]{A}\sqrt[v]{B}$$

holds for all positive real numbers $b_1, \ldots, b_n$ *such that*

$$\sum_{i=1}^{n} b_i^v \leqslant B$$

LJAPUNOV[11] proves the next theorem.

THEOREM 16 (*Ljapunov's inequality*). *Let* x, y *be two vectors of n positive real numbers and* ω *be a weight vector. If* r, s, t *are positive real numbers such that* $r > s > t$,

$$\langle \mathrm{x}, \mathrm{y}^s\rangle^{r-t} \leqslant \langle \mathrm{x}, \mathrm{y}^t\rangle^{r-s} \cdot \langle \mathrm{x}, \mathrm{y}^r\rangle^{s-t}$$

Note that *Generalized Hölder's Inequality* is a generalization of *Cauchy-Bunyakovsky-Schwarz inequality*. We can even prove the *Arithmetic-Geometric Inequality* from *Generalized Hölder's Inequality* in the following way.

$$\frac{a_1 + \ldots + a_n}{n} \geqslant \sqrt[n]{a_1 \cdots a_n}$$
$$\iff (a_1 + \ldots + a_n)^n \geqslant n^n a_1 \cdots a_n$$

This follows from *Generalized Hölder's Inequality* if we set $\mathrm{x}_i = \mathrm{x}$ and $\omega = (\frac{1}{n}, \ldots, \frac{1}{n})$. MINKOWSKI[12] proves the following result.

THEOREM 17 (*Minkowski's Inequality*). *Let* $\mathrm{x}_1, \ldots, \mathrm{x}_n$ *be vectors of n positive real numbers,* ω *be a weight vector and* $r \neq 1$ *be a real number. Then*

$$\mathfrak{M}_r\left(\sum_{i=1}^{n} \mathrm{x}_i, \omega\right) \begin{cases} \geqslant \sum_{i=1}^{n} \mathfrak{M}_r(\mathrm{x}_i, \omega) & \text{if } r < 1 \\ \leqslant \sum_{i=1}^{n} \mathfrak{M}_r(\mathrm{x}_i, \omega) & \text{otherwise} \end{cases}$$

1.7 Rearrangement and Chebyshev Inequalities

Two sequences (a_n) and (b_n) are *similarly sorted* if and only for any i and j, we have

$$(a_i - a_j)(b_i - b_j) \geqslant 0$$

(a_n) and (b_n) are *oppositely sorted* if the inequality is reversed.

[11] ALEKSANDR MICHAJLOVIČ LJAPUNOV

1901 "Nouvelle forme du théorème sur la limite de probabilité", *"Mémoires de l'Académie impériale des sciences de Saint-Pétersbourg*, vol. Série VIII, Tome. 12, `http://www.sudoc.fr/170960269`.

[12] MINKOWSKI, *Geometrie der Zahlen* cit., Page 115 – 117.

Theorem 18 (*Chebyshev inequality*). *Let* (a_n) *and* (b_n) *be two similarly sorted sequences. Then*

$$(\frac{1}{n}\sum_{i=1}^{n} a_i)\cdot(\frac{1}{n}\sum_{i=1}^{n} b_i) \leqslant \frac{1}{n}\sum_{i=1}^{n} a_i b_i$$

Sasser and Slater[13] shows the following result that gives a necessary and sufficient condition for Chebyshev inequality to hold for two sequences (a_n) and (b_n).

Theorem 19 (*Sasser's theorem*).

Seitz[14] proved the following result which combines Cauchy-Schwarz and Chebyshev inequality.

Theorem 20 (*Seitz inequality*). *Let* $(x_n), (y_n), (z_n), (u_n)$ *be sequences of real numbers and* a_{ij} *be a real number for* $1 \leqslant i, j \leqslant n$. *If for all* $i < j$ *and* $r < s$, *the conditions*

$$\begin{vmatrix} x_i & x_j \\ y_i & y_j \end{vmatrix} \cdot \begin{vmatrix} z_r & z_s \\ u_r & u_s \end{vmatrix} \geqslant 0 \text{ and}$$

$$\begin{vmatrix} a_{ri} & a_{rj} \\ a_{si} & a_{sj} \end{vmatrix} \geqslant 0$$

hold, then we have

$$\frac{\sum_{i,j=1}^{n} a_{ij} x_i z_j}{\sum_{i,j=1}^{n} a_{ij} x_i u_j} \geqslant \frac{\sum_{i,j=1}^{n} a_{ij} y_i z_j}{\sum_{i,j=1}^{n} y_i u_j}$$

1.8 Convexity, Jensen and Popoviciu's Inequality

A function $f : I \to \mathbb{R}$ is called *Jensen convex* on some interval $I = [a, b]$ if for any $x, y \in I$,

$$f(\frac{x+y}{2}) \leqslant \frac{f(x)+f(y)}{2}$$

f is *Jensen concave* if $-f$ is Jensen convex. Similarly, a sequence of real numbers (a_n) is *Jensen convex* if

$$2a_n \leqslant a_{n-1} + a_{n+1}$$

holds for $n \in \mathbb{N}$. This concept of convexity was formally introduced first by Jensen.[15] Jensen also proved the following result.

[13] D. w. Sasser and M.L. Slater

1967 "On the inequality $\sum x_i y_i \geqslant \frac{1}{n}\sum x_i \sum y_i$ and the van der Waerden permanent conjecture", *Journal of Combinatorial Theory*, vol. iii, no. 1, pp. 25-33, doi: 10.1016/s0021-9800(67)80012-7.

[14] Jiřé Seitz

1936 "Une remarque aux inégalités", *Aktuárské vědy*, vol. vi, no. 4, pp. 167-171.

[15] Johan Ludwig William Valdemar Jensen

1905 "On konvexe funktioner og uligheder mellem middlvaerdier", *Nyt. Tidsskr. for Math.*, vol. 16B, pp. 49-69; Johan Ludwig William Valdemar Jensen

1906 "Sur les fonctions convexes et les inégalités entre les valeurs moyennes", *Acta Math.*, vol. xxx, pp. 175-193, doi: 10.1007/bf02418571.

THEOREM 21. *Let $w_1, \dots, w_n$ be non-negative rational numbers in the interval I such that $w_1 + \dots + w_n = 1$ and f be a Jensen convex function on I. Then*

$$f(w_1x_1 + \dots + w_nx_n) \leqslant w_1f(x_1) + \dots + w_nf(x_n)$$

But the idea itself was not completely new. HÖLDER[16] proved Theorem 21 under the condition that f can be differentiated at least twice and $f''(x) \geqslant 0$. Ironically, this is exactly the convexity condition from calculus point of view.

Let f be a convex function. For a vector x, let $f(\mathrm{x})$ be the vector $(f(x_1), \dots, f(x_n))$.

THEOREM 22 (*Weighted Jensen's inequality*). *Let f be a convex function. For a vector x of real numbers in the domain of f and a weight vector ω,*

$$f(\mathfrak{A}(\mathrm{x}, \omega)) \leqslant \mathfrak{A}(f(\mathrm{x}), \omega)$$

THEOREM 23 (*Petrovic's Inequality*). *Let f be a convex function and (x_n) be a sequence of positive real numbers. Then*

$$f(x_1) + \dots + f(x_n) \leqslant f(x_1 + \dots + x_n) + (n-1)f(0)$$

THEOREM 24 (*Huygen's inequality*). *Let x be a vector with n elements and ω be a weight vector. Then*

$$\prod_{i=1}^{n}(1 + x_i) \geqslant \mathfrak{G}(\mathrm{x})$$

Note that this is just a special case of *Hölder's inequality*.

THEOREM 25 (*Popoviciu's inequality*). *Let f be a convex function in the interval $[a, b]$. For any $x, y, z \in [a, b]$,*

$$f\left(\frac{x+y+z}{3}\right) + \frac{f(x) + f(y) + f(z)}{3} \geqslant \left(f\left(\frac{x+y}{2}\right) + f\left(\frac{y+z}{2}\right) + f\left(\frac{z+x}{2}\right)\right)$$

[16] HÖLDER, "Ueber einen Mittelwertsatz" cit.

CHAPTER 2.

TRADITIONAL PRINCIPLES

2.1 Engel Form of Cauchy-Schwarz

We mentioned earlier that there is a particular case of Cauchy-Schwarz inequality which is very useful for solving problems. It is known as *Engel form of Cauchy-Schwarz.* Some people also call it *Titu's lemma* or *T2's lemma.* This name became popular among the USA students who attended the IMO training camp after a lecture given by Titu Andreescu at Math Olympiad Summer Program (MOSP) at Georgetown University in June, 2001. Even though it is a direct consequence of Cauchy-Schwarz inequality, it can be proven independently as well. Conversely, Cauchy-Schwarz inequality can be proven using this result.

Theorem 26 (*Engel form of Cauchy-Schwarz). Let $a_1, \dots, a_n, b_1, \dots, b_n$ be real numbers. Then*

$$\frac{a_1^2}{x_1} + \dots + \frac{a_n^2}{x_1} \geqslant \frac{(a_1 + \dots + a_n)^2}{x_1 + \dots + x_n} \tag{2.1}$$

Some writers such as Bellman and Mitrinovitch[1] also call this *Bergström's inequality* due to Bergström.[2]

Proof using Cauchy-Schwarz. Use Cauchy-Schwarz inequality on

$$\frac{a_1}{\sqrt{x_1}}, \dots, \frac{a_n}{\sqrt{x_n}}; \sqrt{x_1}, \dots, \sqrt{x_n}$$

We get

$$\begin{aligned}\left(\frac{a_1^2}{\sqrt{x_1}^2} + \dots + \frac{a_n^2}{\sqrt{x_n}^2}\right)\left(\sqrt{x_1}^2 + \dots + \sqrt{x_n}^2\right) &\geqslant \left(\frac{a_1}{\sqrt{x_1}}\sqrt{x_1} + \dots + \frac{a_n}{\sqrt{x_n}}\sqrt{x_n}\right)^2 \\ &= (a_1 + \dots + a_n)^2\end{aligned}$$

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[1]Richard Bellman

1955 "Notes on Matrix Theory–IV (An Inequality Due to Bergstrom)", *The American Mathematical Monthly*, vol. LXII, no. 3, p. 172, DOI: 10.2307/2306621; D. S. Mitrinovitch

1959 "Equivalence of two sets of inequalities", *The Mathematical Gazette*, vol. XLIII, no. 344, pp. 126-126, DOI: 10.2307/3610201.

[2]H Bergström

1949 "A triangle-inequality for matrices", *Den Elfte Skandinaviske Matematikerkongress, Trondheim*, pp. 264-267.

This proves the theorem. □

Proof by induction. We will use induction to prove 2.1. The inequality is trivial for $n = 1$. For $n = 2$,

$$\begin{aligned} \frac{a^2}{x} + \frac{b^2}{y} &\geqslant \frac{(a+b)^2}{x+y} \\ \iff \frac{a^2y + b^2x}{xy} &\geqslant \frac{a^2 + 2ab + b^2}{x+y} \\ \iff a^2xy + b^2x^2 + a^2y^2 + b^2xy &\geqslant a^2xy + 2abxy + b^2xy \\ \iff (ay - bx)^2 &\geqslant 0 \end{aligned}$$

This is obviously true. Now, assume that the claim is true for □

We get a very useful result as a corollary of this result.

Theorem 27. *Let $a_1, \ldots, a_n, b_1, \ldots, b_n$ be real numbers. Then*

$$\frac{a_1^2}{b_1^2} + \ldots + \frac{a_n^2}{b_n^2} \geqslant \left(\frac{a_1 + \ldots + a_n}{b_1 + \ldots + b_n}\right)^2$$

Proof. By *Engel form of Cauchy-Schwarz*,

$$\begin{aligned} \frac{a_1^2}{b_1^2} + \ldots + \frac{a_n^2}{b_n^2} &\geqslant \frac{(a_1 + \ldots + a_n)^2}{b_1^2 + \ldots + b_n^2} \\ &\geqslant \frac{(a_1 + \ldots + a_n)^2}{(b_1 + \ldots + b_n)^2} \end{aligned}$$

The last inequality follows from the fact that $b_1^2 + \ldots + b_n^2 \leqslant (b_1 + \ldots + b_n)^2$. □

Another result that is similar to *Engel form of Cauchy-Schwarz* is a special case of the *Beckenbach inequality*.

Theorem 28. *Let $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ be positive real numbers. Then*

$$\frac{\sum_{i=1}^{n} x_i^2}{\sum_{i=1}^{n} x_i} + \frac{\sum_{i=1}^{n} y_i^2}{\sum_{i=1}^{n} y_i} \geqslant \frac{\sum_{i=1}^{n} (x_i + y_i)^2}{\sum_{i=1}^{n} (x_i + y_i)}$$

We will start with another proof of *Nesbitt's inequality* with the *Engel form of Cauchy-Schwarz*.

Nesbitt inequality using Engel form. We will use the fact that $a^2+b^2+c^2 \geqslant ab+bc+ca$.

$$\begin{aligned}
S &= \frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \\
&= \frac{a^2}{ab+ca}+\frac{b^2}{bc+ab}+\frac{c^2}{ca+bc} \\
&\geqslant \frac{(a+b+c)^2}{ab+ca+bc+ab+ca+bc} \\
&= \frac{(a+b+c)^2}{2(ab+bc+ca)} \\
&= \frac{a^2+b^2+c^2+2(ab+bc+ca)}{ab+bc+ca} \\
&= \frac{a^2+b^2+c^2}{2(ab+bc+ca)}+1 \\
&\geqslant \frac{1}{2}+1=\frac{3}{2}
\end{aligned}$$

□

Let us use this inequality to solve some problems.

Problem 2.1. Let n be a positive integer and $a_1,\dots,a_n; b_1,\dots,b_n$ be positive real numbers. Show that

$$(a_1b_1+\dots+a_nb_n)\left(\frac{a_1}{b_1}+\dots+\frac{a_n}{b_n}\right) \geqslant (a_1+\dots+a_n)^2$$

Solution. By *Engel form of Cauchy-Schwarz*,

$$\begin{aligned}
\frac{a_1}{b_1}+\dots+\frac{a_n}{b_n} &= \frac{a_1^2}{a_1b_1}+\dots+\frac{a_n^2}{a_nb_n} \\
&\geqslant \frac{(a_1+\dots+a_n)^2}{a_1b_1+\dots+a_nb_n} \\
\iff (a_1b_1+\dots+a_nb_n)\left(\frac{a_1}{b_1}+\dots+\frac{a_n}{b_n}\right) &\geqslant (a_1+\dots+a_n)^2
\end{aligned}$$

Remark. In order to apply the Engel form, we often need transformations like $a/b = a^2/b^2$. Sometimes we will divide the odd power into an even one and send the remaining one in the denominator, for example,

$$\frac{a^{2n+1}}{b} = \frac{(a^n)^2}{\frac{b}{a}}$$

so that we can then apply it. We will demonstrate these ideas in the following problems.

Problem 2.2. For positive real numbers a,b,c, prove that

$$\frac{a}{b+2c}+\frac{b}{c+2a}+\frac{c}{a+2b} \geqslant 1$$

Solution. We again resort to the same trick.

$$\begin{aligned}\frac{a}{b+2c}+\frac{b}{c+2a}+\frac{c}{a+2b} &= \frac{a^2}{ab+2ca}+\frac{b^2}{bc+2ab}+\frac{c^2}{ca+2bc}\\ &\geqslant \frac{(a+b+c)^2}{3(ab+bc+ca)}\\ &= \frac{a^2+b^2+c^2+2(ab+bc+ca)}{3(ab+bc+ca)}\\ &\geqslant \frac{ab+bc+ca+2(ab+bc+ca)}{3(ab+bc+ca)}\end{aligned}$$

In the last line, we used $a^2+b^2+c^2 \geqslant ab+bc+ca$ which is well known by now.

Problem 2.3. Let x, y, z be positive real numbers. Prove that

$$\frac{2}{x+y}+\frac{2}{y+z}+\frac{2}{z+x} \geqslant \frac{9}{x+y+z}$$

Solution. This is again a similar problem.

$$\begin{aligned}\frac{2}{x+y}+\frac{2}{y+z}+\frac{2}{z+x} &= \frac{4}{2(x+y)}+\frac{4}{2(y+z)}+\frac{4}{2(z+x)}\\ &\geqslant \frac{(2+2+2)^2}{2(x+y+y+z+z+x)}\\ &= \frac{36}{4(x+y+z)}\\ &= \frac{9}{x+y+z}\end{aligned}$$

Problem 2.4. Let x, y, z be positive real numbers. Prove that

$$\frac{x^2}{(x+z)(x+y)}+\frac{y^2}{(y+x)(y+z)}+\frac{z^2}{(z+y)(z+x)} \geqslant \frac{3}{4}$$

Solution. First, see that $(x+z)(x+y) = x^2+xy+zx+yz$. Next, we apply *Engel form of Cauchy-Schwarz*.

$$\begin{aligned}
&\frac{x^2}{(x+z)(x+y)} + \frac{y^2}{(y+x)(y+z)} + \frac{z^2}{(z+y)(z+x)} \\
&= \frac{x^2}{x^2+xy+zx+yz} + \frac{y^2}{y^2+yz+xy+zx} + \frac{z^2}{z^2+zx+yz+xy} \\
&\geqslant \frac{(x+y+z)^2}{x^2+y^2+z^2+3(xy+yz+zx)} \\
&= \frac{3}{4}\left(\frac{4(x+y+z)^2}{3(x^2+y^2+z^2+2(xy+yz+zx)+xy+yz+zx)}\right) \\
&= \frac{3}{4}\left(\frac{3(x+y+z)^2+(x+y+z)^2}{3((x+y+z)^2+xy+yz+zx)}\right) \\
&= \frac{3}{4}\left(\frac{3(x+y+z)^2+x^2+y^2+z^2+2(xy+yz+zx)}{3(x+y+z)^2+3(xy+yz+zx)}\right) \\
&\geqslant \frac{3}{4}\left(\frac{3(x+y+z)^2+3(xy+yz+zx)}{3(x+y+z)^2+3(xy+yz+zx)}\right) \\
&= \frac{3}{4}\cdot 1 = \frac{3}{4}
\end{aligned}$$

Thus, the inequality is proved.

Problem 2.5 (IMO Shortlist 1993). For positive real numbers a, b, c and d, prove that

$$\frac{a}{b+2c+3d} + \frac{b}{c+2d+3a} + \frac{c}{d+2a+3b} + \frac{d}{a+2b+3c} \geqslant \frac{2}{3}$$

Solution. Writing

$$\frac{a}{b+2c+3d} = \frac{a^2}{ab+2ca+3ad}$$

and applying *Engel form of Cauchy-Schwarz*, we get

$$\begin{aligned}
S &= \frac{a}{b+2c+3d} + \frac{b}{c+2d+3a} + \frac{c}{d+2a+3b} + \frac{d}{a+2b+3c} \\
&= \frac{a^2}{ab+2ca+3ad} + \frac{b^2}{bc+2bd+3ab} + \frac{c^2}{cd+2ca+3bc} + \frac{d^2}{ad+2bd+3cd} \\
&\geqslant \frac{(a+b+c+d)^2}{4ab+4bc+4ca+4ad+4bd+4cd} \\
&= \frac{2}{3}\left(\frac{3(a+b+c+d)^2}{8(ab+bc+ca+ad+bd+cd)}\right) \\
&= \frac{2}{3}\left(\frac{3(a+b+c+d)^2}{4\left((a+b+c+d)^2-(a^2+b^2+c^2+d^2)\right)}\right) \\
&= \frac{2}{3}\left(\frac{4(a+b+c+d)^2-(a+b+c+d)^2}{4(a+b+c+d)^2-4(a^2+b^2+c^2)}\right)
\end{aligned}$$

By *Cauchy-Bunyakovsky-Schwarz inequality*,

$$\begin{aligned}
&(a^2+b^2+c^2+d^2)(1^2+1^2+1^2+1^2) \geqslant (a+b+c+d)^2 \\
\iff& -4(a^2+b^2+c^2+d^2) \leqslant -(a+b+c+d)^2 \\
\iff& 4(a+b+c+d)^2-4(a^2+b^2+c^2+d^2) \leqslant 4(a+b+c+d)^2-(a+b+c+d)^2
\end{aligned}$$

Thus, we have $S \geqslant \frac{2}{3} \cdot 1 = \frac{2}{3}$.

Problem 2.6 (IMO 1995, problem 2). Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geqslant \frac{3}{2}$$

Solution. Write

$$\frac{1}{a^3(b+c)} = \frac{\frac{1}{a^2}}{a(b+c)}$$

and apply *Engel form of Cauchy-Schwarz*.

$$\begin{aligned}
\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} &= \frac{\frac{1}{a^2}}{a(b+c)} + \frac{\frac{1}{b^2}}{b(c+a)} + \frac{\frac{1}{c^2}}{c(a+b)} \\
&\geqslant \frac{(\frac{1}{a} + \frac{1}{b} + \frac{1}{c})^2}{ab + ca + bc + ab + ca + bc} \\
&= \frac{(\frac{ab+bc+ca}{abc})^2}{2(ab+bc+ca)} \\
&= \frac{ab+bc+ca}{2} \\
&\geqslant \frac{3\sqrt[3]{ab \cdot bc \cdot ca}}{2} \\
&= \frac{3\sqrt[3]{(abc)^2}}{2} \\
&= \frac{3}{2}
\end{aligned}$$

Here, we get $ab + bc + ca \geqslant 3\sqrt[3]{ab \cdot bc \cdot ca}$ by *Arithmetic-Geometric Inequality*.

Problem 2.7 (Tournament of the Towns, 1998). Let a, b, c be positive real numbers. Prove that

$$\frac{a^3}{a^2+ab+b^2} + \frac{b^3}{b^2+bc+c^2} + \frac{c^3}{c^2+ca+a^2} \geqslant \frac{a+b+c}{3}$$

Solution. We again make the numerator a square using

$$\frac{a^3}{a^2+ab+b^2} = \frac{a^4}{a^3+a^2b+ab^2}$$

Then

$$\begin{aligned}
\frac{a^3}{a^2+ab+b^2} + \frac{b^3}{b^2+bc+c^2} + \frac{c^3}{c^2+ca+a^2} &= \frac{a^4}{a^3+a^2b+ab^2} + \frac{b^4}{b^3+b^2c+bc^2} + \frac{c^4}{c^3+c^2a+ca^2} \\
&\geqslant \frac{(a^2+b^2+c^2)^2}{a^3+b^3+c^3+ab(a+b)+bc(b+c)+ca(c+a)} \\
&= \frac{(a^2+b^2+c^2)^2}{(a+b+c)(ab+bc+ca)} \\
&= \frac{a^2+b^2+c^2}{a+b+c}
\end{aligned}$$

The last line follows since $a^2 + b^2 + c^2 \geqslant ab + bc + ca$. Then the conclusion follows because by *Cauchy-Bunyakovsky-Schwarz inequality*,

$$\begin{aligned}(a^2 + b^2 + c^2)(1^2 + 1^2 + 1^2) &\geqslant (a + b + c)^2 \\ \iff \frac{a^2 + b^2 + c^2}{a + b + c} &\geqslant \frac{a + b + c}{3}\end{aligned}$$

2.2 The Buffalo Way

The buffalo way is a *bashing* method of solving inequality problems. It usually exploits symmetry or cyclic property and involves tedious calculations, so it is not always elegant. Nonetheless it is a useful tool. The idea is to assume something like $a \leqslant b \leqslant c$ without losing any generality, specially when there is symmetry. In general, we want to assume that

$$\begin{aligned}&a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n \\ &a_2 = a_1 + b_1 \\ &\vdots \\ &a_n = a_1 + b_n\end{aligned}$$

We can also assume the following.

$$\begin{aligned}a_2 &= a_1 + c_1 \\ a_3 &= a_2 + c_2 \\ &= a_1 + c_1 + c_2 \\ &\vdots \\ a_n &= a_{n-1} + c_{n-1} \\ &= a_1 + c_1 + \ldots + c_{n-1}\end{aligned}$$

where $c_1, \ldots, c_{n-1} \geqslant 0$. Out of these two forms, the latter is actually a consequence of the former one and is usually the more used one. We will see a few problems that implement this technique.

Problem 2.8. Prove that for two non-negative reals x, y

$$x + y \geqslant 2\sqrt{xy}$$

Solution. Since the inequality is symmetric on x and y, we can assume without loss of generality that $x \leqslant y$. Let $y = x + k$.

$$\begin{aligned}x + y &\geqslant 2\sqrt{xy} \\ \iff 2x + k &\geqslant 2\sqrt{x(x + k)} \\ \iff 4x^2 + 4xk + k^2 &\geqslant 4x^2 + 4xk \\ \iff k^2 &\geqslant 0\end{aligned}$$

which is evident.

Try proving the case $n = 3$ of arithmetic-geometric mean inequality with this technique. You will understand why we said this is often a tedious technique.

Problem 2.9.

$$x + y + z \geqslant 3\sqrt[3]{xyz}$$

Let us also prove the following using buffalo way.

Problem 2.10. Let $a, b, c > 0$ be real numbers. Prove that

$$a^2 + b^2 + c^2 \geqslant ab + bc + ca$$

Solution. Assume that $a \leqslant b \leqslant c$ and $b = a + x, c = a + x + y$ where $x, y \geqslant 0$.

$$\begin{aligned} a^2 + (a+x)^2 + (a+x+y)^2 &\geqslant a(a+x) + (a+x)(a+x+y) + (a+x+y)a \\ \iff 3a^2 + 2x^2 + y^2 + 4ax + 2xy + 2ay &\geqslant 3a^2 + x^2 + 4ax + 2ay + 2xy \\ \iff x^2 + y^2 &\geqslant 0 \end{aligned}$$

This is again true.

Problem 2.11. Let $a, b, c \geqslant 0$ be real numbers such that $a \leqslant b \leqslant c$. Prove that

$$(a+b)(c+a)^2 \geqslant 6abc$$

Solution. Since $a \leqslant b \leqslant c$, let $b = a + x, c = a + x + y$.

$$\begin{aligned} (2a+x)(2a+x+y)^2 &\geqslant 6a(a+x)(a+x+y) \\ \iff (2a+x)(4a^2 + x^2 + y^2 + 4ax + 4ay + 2xy) &\geqslant 6a(a^2 + 2ax + x^2 + ay + xy) \\ \iff 8a^3 + 12a^2x + 8a^2y + 6ax^2 + 8axy + 2ay^2 + x^3 + 2ax^2 + xy^2 &\geqslant 6a^3 + 12a^2 + 6ax^2 + 6a^2y + 6axy \\ \iff 2a^3 + 2a^2y + 2axy + 2ay^2 + x^3 + 2x^2y + xy^2 &\geqslant 0 \end{aligned}$$

This evidently true.

Now, we will prove Nesbitt's inequality using the first form of the buffalo way.

Proof of Nesbitt's inequality by the buffalo way. Let $x \leqslant y \leqslant z$ and $y = x + a, z = y + b$ with $a \geqslant 0, b \geqslant 0$. We are required to prove that

$$\frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} \geqslant \frac{3}{2}$$

Let us express each variable as a parameter of a single variable.

$$\begin{aligned} \frac{y-a}{2y+b} + \frac{y}{2y+b-a} + \frac{y+b}{2y-a} &\geqslant \frac{3}{2} \\ \iff \frac{y-a}{2y+b} - \frac{1}{2} + \frac{y}{2y+b-a} - \frac{1}{2} + \frac{y+b}{2y-a} - \frac{1}{2} &\geqslant 0 \\ \iff \frac{1}{2}\left(\frac{2y-2a-2y-b}{2y+b} + \frac{2y-2y-b+a}{2y+b-a} + \frac{2y+2b-2y+a}{2y-a}\right) &\geqslant 0 \\ \iff -\frac{2a+b}{2y+b} + \frac{a-b}{2y+b-a} + \frac{2b+a}{2y-a} &\geqslant 0 \end{aligned}$$

The last inequality actually holds because of the following:

$$\begin{aligned}\frac{2b+a}{2y-a}+\frac{a-b}{2y+b-a}-\frac{2a+b}{2y+b} &\geqslant \frac{2b+a}{2y+b-a}+\frac{a-b}{2y+b-a}-\frac{2a+b}{2y+b}\\ &\geqslant \frac{2a+2b}{2y+b-a}-\frac{2a+b}{2y+b}\\ &\geqslant \frac{2a+2b}{2y+b}-\frac{2a+b}{2y+b}\\ &=\frac{a}{2y+b}\end{aligned}$$

which is obviously at least 0. Here, we have used the facts that $\frac{1}{x}\geqslant\frac{1}{y}$ implies $\frac{1}{x-u}\geqslant\frac{1}{y}$ and $\frac{1}{x}\geqslant\frac{1}{y+v}$ for $u,v\geqslant 0$. □

The next inequality is a nice application of the buffalo way.

PROBLEM 2.12. Let $x,y,z>0$ be real numbers such that no two are equal. Show that,

$$\frac{1}{(x-y)^2}+\frac{1}{(y-z)^2}+\frac{1}{(z-x)^2}\geqslant\frac{4}{xy+yz+zx}$$

Solution. The expression on the left is symmetric on x,y,z. Without loss of generality, assume that $x\leqslant y\leqslant z$ and $y=x+a, z=x+a+b$.

$$\begin{aligned}\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{(a+b)^2} &\geqslant \frac{4}{x(x+a)+(x+a)(x+a+b)+(x+a+b)x}\\ &=\frac{4}{x^2+xa+x^2+2ax+a^2+bx+ab+x^2+xa+bx}\\ &\geqslant\frac{4}{3x^2+4xa+2xb+ab+a^2}\end{aligned}$$

Note that there is no x on the left side of the inequality. So, we can use $x>0$ to get rid of it.

$$3x^2+4xa+2xb+ab+a^2>ab+a^2$$
$$\frac{4}{3x^2+4xa+2xb+ab+a^2}<\frac{4}{ab+a^2}$$

So, we are done if we can prove the following.

$$\begin{aligned}\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{(a+b)^2} &\geqslant \frac{4}{ab+a^2}\\ &=\frac{4}{a(b+a)}\end{aligned}$$

This is not a trivial inequality. We have to show that this actually holds.

$$\begin{aligned}\frac{1}{a^2}+\frac{1}{b^2}-\frac{2}{a(b+a)}+\frac{1}{(a+b)^2} &\geqslant \frac{2}{a(b+a)}\\ \iff \frac{1}{b^2}+(\frac{1}{a}-\frac{1}{a+b})^2 &\geqslant \frac{2}{a(a+b)}\\ \iff \frac{1}{b^2}+(\frac{b}{a(a+b)})^2 &\geqslant \frac{2}{a(a+b)}\\ \iff \frac{1}{b^2}+\frac{b^2}{a^2(a+b)^2} &\geqslant \frac{2}{a(a+b)}\end{aligned}$$

The last inequality is true by arithmetic-geometric mean inequality since

$$\begin{aligned}\frac{1}{b^2}+\frac{b^2}{a^2(a+b)^2} &\geqslant 2\sqrt{\frac{1}{b^2}\cdot\frac{b^2}{a^2(a+b)^2}}\\ &= \frac{2}{a(a+b)}\end{aligned}$$

Solve the next problem as an exercise.

Problem 2.13. For all real $x, y, z \geqslant 0$, prove that

$$\frac{x^3+y^3+z^3}{x^2y+y^2z+z^2x} \geqslant \sum \frac{x}{y+z} - 1$$

2.3 Majorization and Symmetric Sums

Let x and y be two vectors of n real numbers. We say that x *dominates* or *majorizes* y if

$$\begin{aligned} x_1 &\geqslant x_2 \geqslant \ldots \geqslant x_n\\ y_1 &\geqslant y_2 \geqslant \ldots \geqslant y_n\\ x_1+\ldots+x_n &= y_1+\ldots+y_n\\ x_1+\ldots+x_k &\geqslant y_1+\ldots+y_k\end{aligned}$$

for $1 \leqslant k \leqslant n-1$. If x dominates y (resp. y is *dominated by* x), then we write x $\succ$ y (resp. y $\prec$ x). For example, $(4,0,0) \succ (3,1,0) \succ (2,2,0)$. The vectors x and y need not be monotonic because we can just sort them into monotonic vectors.

We will also introduce the cyclic and symmetric polynomials and notations juxtaposed with them. The expression $x^2+y^2+z^2$ is *symmetric* whereas $x^2y+y^2z+z^2x$ is *cyclic* but not symmetric because $y^2x+z^2y+x^2z \neq x^2y+y^2z+z^2x$. A symmetric polynomial in the variables $x_1,\ldots,x_n$ should remain same regardless of the order in which the variables are used. So $f(x_1,\ldots,x_n)$ is symmetric if f remains *invariant* for all permutations of $x_1,\ldots,x_n$ in the expression unlike the cyclic example we just saw. For example, $xy+yz+zx$ is symmetric and so is xyz. And $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}$ is cyclic but not symmetric. We can use the cyclic and symmetric notations to represent the expressions in a short form. Here are some demonstrations.

$$\begin{aligned} a^2+b^2+c^2 &= \sum_{cyc} a^2\\ a^2b+b^2c+c^2a &= \sum_{cyc} a^2b\\ xy+yz+zx &= \sum_{cyc} xy\end{aligned}$$

Note that even though the expression $xy+yz+zx$ and $a^2+b^2+c^2$ are symmetric, we do not consider them symmetric polynomial sums in this notation. A symmetric polynomial sum should have all $n!$ terms in the sum since it is symmetric on all $n!$ permutations of the variables in it. Even if there can be duplicates, the total number of terms should still remain $n!$. For this reason, this sum is often denoted by

$$\sum ! \text{ or } \sum_{\sigma}$$

where σ indicates that the sum is taken over all possible permutations. Here are some examples.

$$\sum_{sym} a^2 = 2(a^2 + b^2 + c^2)$$
$$\sum ! x^2y = x^2y + y^2x + y^2z + z^2y + z^2x + x^2z$$
$$\sum_{\sigma} x^3y = x^3y + x^3z + y^3z + y^3x + z^3x + z^3y$$

Let x and a be vectors with n elements. We write

$$\begin{aligned} F(\mathrm{x}, \mathrm{a}) &= x_1^{a_1} \cdots x_n^{a_n} \\ T(\mathrm{x}, \mathrm{a}) &= \sum ! F(\mathrm{x}, \mathrm{a}) \\ &= \sum_{sym} F(\mathrm{x}, \mathrm{a}) \\ &= \sum ! x_1^{a_1} \cdots x_n^{a_n} \end{aligned}$$

We will explain it a little bit more. Say, we want to find the symmetric polynomials of the type x^3y for the variables x, y, z. Write it as $x^3y^1z^0$. Now, we fix the powers $3, 1, 0$ in their respective places and let x, y, z run through the permutations. Then we get

$$x^3y^1z^0 + x^3z^1y^0 + y^3z^1x^0 + y^3x^1z^0 + z^3x^1y^0 + z^3y^1x^0$$

See that this is exactly what we got in the last line of

$$\sum_{\sigma} x^3y$$

Some authors write this notation as

$$T[\mathrm{a}](\mathrm{x}) = T[a_1, \dots, a_n](x_1, \dots, x_n)$$

Simply $T[\mathrm{a}] = T[a_1, \dots, a_n]$ is used if it is clear what x is.

$$T[1, 0, \dots, 0] = (x_1 + \dots + x_n)(n-1)!$$
$$T\left[\frac{1}{n}, \dots, \frac{1}{n}\right] = n! \sqrt[n]{x_1 \cdots x_n}$$

Then for a vector of positive real numbers x, we can write the arithmetic-geometric mean inequality as

$$T[1, 0, \dots, 0] \geqslant T\left[\frac{1}{n}, \dots, \frac{1}{n}\right]$$

We can also define mean values based on the symmetric polynomials. We call

$$\mathfrak{M}[\mathrm{a}](\mathrm{x}) = \frac{T[\mathrm{a}](\mathrm{x})}{n!}$$

the *symmetrical mean*. For example,

$$\begin{aligned}\mathfrak{M}[1,0\dots,0] &= \frac{(n-1)!}{n!}(a_1+\dots+a_n)\\ &= \mathfrak{A}(\mathrm{a})\\ \mathfrak{M}\left[\frac{1}{n},\dots,\frac{1}{n}\right] &= \frac{n!}{n!}a_1^{\frac{1}{n}}\cdots a_n^{\frac{1}{n}}\\ &= \mathfrak{G}(\mathrm{a})\end{aligned}$$

So, the symmetrical mean is a generalization of $\mathfrak{A}$ and $\mathfrak{G}$. While we are talking about symmetric sums, we should consider the next identity.

$$(x+a_1)\cdots(x+a_n) = x^n + \binom{n}{1}x^{n-1}D_1 + \binom{n}{2}x^{n-2}d_2 + \dots + \binom{n}{n}D_n$$

where D_k is the sum of products of $a_1,\dots,a_n$ taken k at the same time. If S_k is the coefficient of x^{n-k} in the expansion, then

$$S_k = \binom{n}{i}D_k$$

Theorem 29 (*Newton's inequality*). *For* $1 \leqslant i \leqslant n-1$, *we have*

$$D_i^2 \geqslant D_{i-1}D_{i+1}$$

Theorem 30 (*Maclaurin's inequality*). *With the same notation as Newton's inequality,*

$$D_1 \geqslant \sqrt{D_2} \geqslant \dots \geqslant \sqrt[n]{D_n}$$

2.4 Karamata's Inequality

Karamata[3] proves the next theorem regarding convex functions when one vector majorizes another.

Theorem 31 (*Karamata's inequality*). *Let f be a convex function and* a *and* b *be two vectors such that* a $\succ$ b. *Then*

$$\sum_{i=1}^{n} f(a_i) \geqslant \sum_{i=1}^{n} f(b_i)$$

The next result first appeared at the forum *Art of Problem Solving*[4]. *Daniel S. Liu* showed this as a separate result but it also follows from *Karamata's inequality*.

Theorem 32 (*Reverse rearrangement inequality*). *Let σ be a permutation of* $\{1,\dots,n\}$ *and* $(a_n),(b_n)$ *be two similarly sorted sequences of positive real numbers. Then*

$$(a_1+b_1)\cdots(a_n+b_n) \leqslant (a_1+b_{\sigma(1)})\cdots(a_n+b_{\sigma(n)}) \leqslant (a_1+b_n)\cdots(a_n+b_1)$$

[3]Jovan Karamata
1932 "Sur une inégalité rélative aux fonctions convexes", French, *Publ. Math. Univ. Belgrade*, vol. I, pp. 145-148.
[4]`https://artofproblemsolving.com/community/q2h617132p3677200`

Later this was generalized to the following result.

Theorem 33 (*Generalized reverse rearrangement inequality*). *Let* $(a_1), \dots, (a_m)$ *be* m *similarly sorted sequences of* n *positive real numbers. Then*

$$\prod_{i=1}^{m} \sum_{j=1}^{n} a_{\sigma_j(i),j} \geqslant \prod_{i=1}^{m} \sum_{j=1}^{n} a_{i,j}$$

2.5 The Bunching Method: Muirhead's Inequality

Muirhead[5] proves a very important result regarding symmetric sums and means. It is widely used for solving problems, specially coupled with some other method such as The Buffalo Way. It is also known as *the bunching principle* among the American students.

Theorem 34 (*Muirhead's inequality*). *Let* x *be a vector of positive real numbers and* a *be dominated by* b. *Then*

$$\mathfrak{M}[\mathrm{a}](\mathrm{x}) \leqslant \mathfrak{M}[\mathrm{b}](\mathrm{x})$$

Paris and Vencovská[6] proves a generalization of this inequality but the result is of little practical use for Olympiad purposes.

Theorem 35 (*Generalization of Muirhead's inequality*). *Let* x *and* y *be two vectors of positive real numbers such that* $\mathrm{x} \succ \mathrm{y}$. *If* $p_1, \dots, p_k$ *are non-negative real numbers, then*

$$\sum_{\substack{S_1 \cup \ldots \cup S_r = \{1,\ldots,k\} \\ S_i \cap S_j = \varphi}} \prod_{j=1}^{r} \Big(\sum_{i \in S_j} p_i\Big)^{\square x_j} \geqslant \sum_{\substack{S_1 \cup \ldots \cup S_r = \{1,\ldots,k\} \\ S_i \cap S_j = \varphi}} \prod_{j=1}^{r} \Big(\sum_{i \in S_j} p_i\Big)^{\square y_j}$$

where $\square$ *in* $\square x_j$ *denotes that in the expansion of this power, we only consider the terms with non-zero power of* p_i.

Problem 2.14. Let a, b, c be positive real numbers. Prove that

$$a^7 + b^7 + c^7 \geqslant a^4b^3 + b^4c^3 + c^4a^3$$

Solution. Since $(7,0,0) \succ (4,3,0)$, by *Muirhead's inequality* we have

$$\begin{aligned} a^7b^0c^0 + a^7c^0b^0 + b^7c^0a^0 + b^7a^0c^0 + c^7a^0b^0 + c^7b^0a^0 &\geqslant a^4b^3c^0 + a^4c^3b^0 + b^4c^3a^0 \\ &\quad + b^4a^3c^0 + c^4a^3b^0 + c^4b^3a^0 \\ \iff 2(a^7 + b^7 + c^7) &\geqslant 2(a^4b^3 + b^4c^3 + c^4a^3) \\ \iff a^7 + b^7 + c^7 &\geqslant a^4b^3 + b^4c^3 + c^4a^3 \end{aligned}$$

[5] R. F. Muirhead
1902 "Some Methods applicable to Identities and Inequalities of Symmetric Algebraic Functions of n Letters", *Proceedings of the Edinburgh Mathematical Society*, vol. xxi, pp. 144-162, doi: 10.1017/s001309150003460x.

[6] J. B. Paris and A. Vencovská
2009 "A Generalization of Muirhead's Inequality", *Journal of Mathematical Inequalities*, no. 2, pp. 181-187, doi: 10.7153/jmi-03-18.

PROBLEM 2.15. Let a, b, c be positive real numbers. Prove that

$$\frac{a^3}{bc} + \frac{b^3}{ca} + \frac{c^3}{ab} \geqslant a + b + c$$

Solution. We can rearrange the inequality into

$$\begin{aligned} &\frac{a^4 + b^4 + c^4}{abc} \geqslant a + b + c \\ \iff\ & a^4 + b^4 + c^4 \geqslant a^2bc + b^2ca + c^2ab \end{aligned}$$

This follows from *Muirhead's inequality* since $(4, 0, 0) \succ (2, 1, 1)$.

2.6 Homogenization and Normalization

2.7 Substitutions

CHAPTER 3.

ADVANCED TECHNIQUES & INEQUALITIES

3.1 Tangent Line Trick

The *tangent line trick* is a very useful trick which has been around for quite some time. Li[1] demonstrates some problems with this trick. Here, we will discuss this in detail and show how it can tackle problems.

Imagine that we are given an inequality in $a_1, \dots, a_n$. The inequality can be divided into a sum of expressions which is same for $a_1, \dots, a_n$. That is it can be written as $f(a_1) + \dots + f(a_n) \geqslant g(a_1, \dots, a_n)$. Now, we typically try to use Jensen's inequality in cases like this. But in many cases, f is not convex. Even so, we may still be able to prove something like

$$f(x) \geqslant f(\bar{a}) + (x - \bar{a})f'(\bar{a})$$

where $\bar{a} = \frac{a_1 + \dots + a_n}{n}$. Summing this for $x = a_1, \dots, a_n$, we have

$$f(a_1) + \dots + f(a_n) \geqslant nf(\bar{a})$$

The motivation sort of comes from basic calculus. By *wishful thinking*, we may be able to prove that the slope between the points $(x, f(x))$ and $(\bar{a}, f(\bar{a}))$ is at least the slope of the tangent line of $f(x)$ at $x = \bar{a}$. This is where the name comes from. This trick is specially useful if you are given the quantity $a_1 + \dots + a_n$. If the expressions involved are homogeneous, then you do not even need this value. You can use homogeneity to impose conditions such as $a + b + c = 1$. For example, see this problem.

Problem 3.1. Let a, b, c be positive real numbers. Prove that

$$\frac{1}{a(b+c)} + \frac{1}{b(c+a)} + \frac{1}{c(a+b)} \geqslant \frac{27}{2(a+b+c)^2}$$

Solution. The inequality is homogeneous in a, b, c. So without loss of generality, we can normalize the inequality assuming that $a + b + c = 1$. Then $0 < a, b, c < 1$, $\bar{a} = 1/3$ and we get the transformation

$$\frac{1}{a(1-a)} + \frac{1}{b(1-b)} + \frac{1}{c(1-c)} \geqslant \frac{27}{2}$$

---

[1] Kin-Yin Li

2006 "Using Tangent Lines to Prove Inequalities", *Mathematical Excalibure*, vol. x, no. 5, p. 1.

We define

$$f(x) = \frac{1}{a(1-a)}$$

and attempt to prove the inequality that is required by tangent line trick. Since

$$f'(x) = \frac{2x-1}{x^2(1-x)^2}$$

we have $f(1/3) = 9/2$ and $f'(1/3) = -27/4$. Thus, if we can prove the inequality

$$f(x) \geqslant \frac{9}{2} - \frac{27}{4}(x - \frac{1}{3})$$

we are done since we will have

$$\begin{aligned} f(a) + f(b) + f(c) &\geqslant 3 \cdot f(\frac{1}{3}) \\ &= \frac{27}{2} \end{aligned}$$

So we should check if the tangent line inequality indeed holds.

$$\begin{aligned} \frac{1}{x(1-x)} &\geqslant \frac{9}{2} - \frac{27}{4}(x - \frac{1}{3}) \\ \iff \frac{1}{x(1-x)} &\geqslant \frac{54 - 81x + 27}{12} \\ &\geqslant \frac{27(1-x)}{4} \\ \iff x(1-x)^2 &\leqslant \frac{4}{27} \end{aligned}$$

We can check if this holds. If $g(x) = x(1-x)^2$, then

$$\begin{aligned} g'(x) &= -2x(1-x) + (1-x)^2 \\ &= (1-x)(1-3x) \end{aligned}$$

We have $g'(x) = 0$ if $x \in \{1, 1/3\}$ and

$$\begin{aligned} g''(x) &= -3(1-x) - (1-3x) \\ &= 6x - 4 \end{aligned}$$

Since $0 < x < 1$ and $g''(1) = 2 > 0$, $g''(1/3) = -2 < 0$, we have that $g(x)$ is maximum at $x = 1/3$ in the interval $(0, 1)$. Thus,

$$\begin{aligned} x(1-x)^2 &\leqslant \frac{1}{3}(1 - \frac{1}{3})^2 \\ &= \frac{27}{4} \end{aligned}$$

We are finally done.

Remark. Tangent line is a neat trick but it may always not be pretty. But before you go all in, you can easily check if the desired inequality follows from

$$f(a_1) + \dots + f(a_n) \geqslant nf(\bar{a})$$

or not. If it does, there is a good chance that $f(x) \geqslant f(\bar{a}) + (x - \bar{a})f'(\bar{a})$ holds as well. Another indication that this might work is that equality occurs for $a_1 = \dots = a_n = \bar{a}$. In order to avoid calculation with fractional values, we could also assume $a_1 + \dots + a_n = n$ for normalization so $\bar{a} = 1$. Also, we may sometimes have to deal with $\leqslant$ instead of $\geqslant$. Let us see an example of this type below.

PROBLEM 3.2. Given positive real numbers a, b, c such that $a + b + c \geqslant 3$. Prove that

$$\frac{1}{a^2+b+c} + \frac{1}{a+b^2+c} + \frac{1}{a+b+c^2} \leqslant 1$$

Solution. We have equality in the case $a = b = c = 1$. So, we may be optimistic that tangent line trick will work here. Using $b + c \geqslant 3 - a$,

$$\frac{1}{a^2+b+c} + \frac{1}{a+b^2+c} + \frac{1}{a+b+c^2} \leqslant \frac{1}{a^2+3-a} + \frac{1}{b^2+3-b} + \frac{1}{c^2+3-c}$$

So we are done if we can prove that

$$\frac{1}{a^2+3-a} + \frac{1}{b^2+3-b} + \frac{1}{c^2+3-c} \leqslant 1$$

Let

$$f(x) = \frac{1}{x^2+3-x}$$

We have that

$$f'(x) = \frac{1-2x}{(x^2+3-x)^2}$$

Since $f(1) = 1/3$ and $f'(1) = -1/9$, we need to prove the tangent line inequality

$$\begin{aligned} f(x) &\leqslant \frac{1}{3} - \frac{1}{9}(x-1) \\ \iff \frac{1}{x^2+3-x} &\leqslant \frac{4-x}{9} \\ \iff 5x^2 + 3 - 7x - x^3 &\geqslant 0 \\ \iff x^3 - 5x^2 + 7x - 3 &\leqslant 0 \end{aligned}$$

We can see that $(x - 3)$ is a factor of this polynomial. So we can factor it easily.

$$\begin{aligned} \iff x^2(x-3) - 2x(x-3) + x - 3 &\leqslant 0 \\ \iff (x-3)(x^2-2x+1) &\leqslant 0 \\ \iff (3-x)(x-1)^2 &\geqslant 0 \end{aligned}$$

This is obviously true. And summing up the tangent line inequality, we get the conclusion.

PROBLEM 3.3 (USAMO 2003). Let a, b, c be positive real numbers. Prove that

$$\frac{(2a+b+c)^2}{2a^2+(b+c)^2}+\frac{(2b+c+a)^2}{2b^2+(c+a)^2}+\frac{(2c+a+b)^2}{2c^2+(a+b)^2}\leqslant 8$$

Solution. Due to homogeneity, we can assume that $0 < a, b, c < 1$ and $a+b+c=1$ without loss of generality. Then we are required to prove

$$\frac{(1+a)^2}{2a^2+(1-a)^2}+\frac{(1+b)^2}{2b^2+(1-b)^2}+\frac{(1+c)^2}{2c^2+(1-c)^2}\leqslant 8$$

We can see that equality occurs for $a=b=c=1/3$. Define

$$\begin{aligned} f(x) &= \frac{(1+x)^2}{2x^2+(1-x)^2} \\ &= \frac{(1+x)^2}{3x^2-2x+1} \end{aligned}$$

The tangent line inequality in this case is

$$f(x) \leqslant f(\bar{a}) + (x-\bar{a})f'(\bar{a})$$

where

$$\begin{aligned} f'(x) &= \frac{2(3x^2-2x+1)(1+x)-2(1+x)^2(3x-1)}{(3x^2-2x+1)^2} \\ &= \frac{2(1+x)}{3x^2-2x+1}-\frac{2(1+x)^2(3x-1)}{(3x^2-2x+1)^2} \end{aligned}$$

We have $f(1/3)=8/3$ and $f'(1/3)=4$ so if we can prove that

$$\begin{aligned} f(x) &\leqslant \frac{8}{3}+4\left(x-\frac{1}{3}\right) \\ &= \frac{4(3x+1)}{3} \end{aligned}$$

for $0 < x < 1$, then the inequality follows.

$$\begin{aligned} & f(x) \leqslant \frac{4(3x+1)}{3} \\ \iff & \frac{(1+x)^2}{3x^2-2x+1} \leqslant \frac{12x+3}{3} \\ \iff & 36x^3-15x^2-2x+1 \geqslant 0 \\ \iff & (3x-1)^2(4x+1) \geqslant 0 \end{aligned}$$

This is obviously true. So, we have the inequality.

3.2 Schur-Vornicu-Mildorf Inequality

Hardy et al.[2] states the following as *Schur's inequality*.

Theorem 36 (Schur's inequality). *Let n be a positive integer. If a, b, c are positive real numbers, then*

$$a^n(a-b)(a-c) + b^n(b-c)(b-a) + c^n(c-a)(c-b) \geqslant 0$$

Barnard and Child[3] states this theorem for $n \leqslant -1$ as well. Hardy et al.[4] mentions this result in connection with the following result.

Theorem 37. *Consider a vector of positive real numbers* x *such that no two elements are equal. If $v \geqslant 0$ and $\delta > 0$, then*

$$\mathfrak{M}[v+2\delta, 0, 0, a_4, \ldots] - 2\mathfrak{M}[v+\delta, \delta, 0, a_4, \ldots] + \mathfrak{M}[v, \delta, \delta, a_4, \ldots] \geqslant 0$$

for a vector of non-negative real numbers a.

It was Issai Schur who informed Hardy et al.[5] that Theorem 37 does not follow from *Muirhead's inequality* but from Schur's inequality if we set $n = \frac{v}{\delta}$. So Theorem 36 is accredited to Schur following Neville, Oppenheim, Watson, and Wright.[6] Schur's inequality and its reverse have been generalized in many ways. For example, Guha[7] proves the following result.

Theorem 38 (Guha's inequality). *Let a, b, c, u, v, w be positive real numbers such that for a real number p,*

$$\sqrt[p]{a} + \sqrt[p]{c} \leqslant \sqrt[p]{b} \tag{3.1}$$

$$\sqrt[p+1]{u} + \sqrt[p+1]{w} \leqslant \sqrt[p+1]{v} \tag{3.2}$$

If $p > 0$, then

$$ubc - vca + wab \geqslant 0 \tag{3.3}$$

[2] Hardy et al., *Inequalities* cit., Page 64.

[3] S. Barnard and J. M. Child

2018 *Higher Algebra*, New Academic Science, Page 217.

[4] Hardy et al., *Inequalities* cit.

[5] Ibid.

[6] E. H. Neville

1956 "Schur's inequality", *The Mathematical Gazette*, vol. xl, no. 333, pp. 216-216, doi: 10.2307/3608826; A. Oppenheim

1958 "2739. Generalisations of Schur's inequality", *The Mathematical Gazette*, vol. xlii, no. 339, pp. 35-35, doi: 10.2307/3608352; G. N. Watson

1956 "A trivial inequality", *The Mathematical Gazette*, vol. xl, no. 334, pp. 288-288, doi: 10.2307/3609631; G.N. Watson

1955 "Schur's Inequality", *The Mathematical Gazette*, vol. xxxix, no. 329, pp. 207-208, doi: 10.2307/3608749; E. M. Wright

1956 "A generalisation of Schur's inequality", *The Mathematical Gazette*, vol. xl, no. 333, pp. 217-217, doi: 10.2307/3608827.

[7] U. C. Guha

1962 "Inequalities leading to generalisations of Schur's inequality", *The Mathematical Gazette*, vol. xlvi, no. 357, pp. 227-229, doi: 10.2307/3614027.

If $-1 < p < 0$*, then 3.3 is reverse. If* $p < -1$*, then 3.1 and 3.2 need to be reversed. Equality occurs if and only if*

$$\frac{a^{p+1}}{u^p} = \frac{b^{p+1}}{v^p} = \frac{c^{p+1}}{w^p}$$

Oppenheim and Davies[8] proves the following result which can be thought as the reverse of Schur's inequality.

Theorem 39. *Let* $n \geqslant 3$ *be an integer and* $a_1, \ldots, a_n$ *be real numbers. Then*

$$\sum_{i=1}^{n}(x_i - x_1)\cdots(x_i - x_{i-1}) \cdot (x_i - x_{i+1})\cdots(x_i - x_n) \geqslant 0$$

holds for all real numbers $x_1, \ldots, x_n$ *such that* $x_1 \geqslant \ldots \geqslant x_n$ *if*

$$\begin{cases} a_1 \geqslant 0; a_2 \leqslant \left(a_1^{\frac{1}{2}} + a_3^{\frac{1}{2}}\right)^2; a_3 \geqslant 0 & \text{if } n = 3 \\ a_2 \leqslant a_1; (-1)^n(a_{n-1} - a_n) \geqslant 0; (-1)^{k+1}a_k \geqslant 0 & \text{if } n \geqslant 4 \end{cases}$$

where $1 \leqslant k \leqslant n-1$ *and* $k \notin \{2, n-1\}$.

We will first mention a notable generalization of Schur's inequality.

Theorem 40 (Schur-Vornicu-Mildorf inequality). *Let* a, b, c *be three real numbers and* x, y, z *be non-negative real numbers. Then*

$$x(a-b)(a-c) + y(b-c)(b-a) + z(c-a)(c-b) \geqslant 0$$

holds if one of the following conditions hold:

(1) $a \geqslant b \geqslant c$ *and* $x \geqslant y$.

(2) $a \geqslant b \geqslant c$ *and* $z \geqslant y$.

(3) $a \geqslant b \geqslant c$ *and* $x + z \geqslant y$.

(4) a, b, c *are non-negative,* $a \geqslant b \geqslant c$ *and* $ax \geqslant by$.

(5) a, b, c *are non-negative,* $a \geqslant b \geqslant c$ *and* $cz \geqslant by$.

(6) a, b, c *are non-negative,* $a \geqslant b \geqslant c$ *and* $ax + cz \geqslant by$.

(7) x, y, z *are lengths of a triangle.*

(8) x, y, z *are square of lengths of a triangle.*

(9) ax, by, cz *are the lengths of a triangle.*

(10) *There is a convex function* f *on the interval* I *such that* $x = f(a), y = f(b), z = f(c)$ *for non-negative real numbers* $a, b, c \in I$.

[8] A. Oppenheim and Roy O. Davies
1964 "Inequalities of Schur's Type", *The Mathematical Gazette*, vol. XLVIII, no. 363, pp. 25-27, DOI: 10.2307/3614303.

Vornicu[9] generalized this result to the following.

Theorem 41 (Vornicu-Schur inequality). *Let $a, b, c; x, y, z$ be real numbers such that $a \geqslant b \geqslant c$ and either $x \geqslant y \geqslant z$ or $z \geqslant y \geqslant x$. Let k be a positive integer f be a non-negative real valued convex or monotonic function. Then*

$$f(x)(a-b)^k(a-c)^k + f(y)(b-c)^k(b-a)^k + f(z)(c-a)^k(c-b)^k \geqslant 0$$

We can easily verify that Schur's inequality follows from Vornicu-Schur inequality by setting $x = a, y = b, z = c, k = 1$ and $f(m) = m^n$.

3.3 Radon's Inequality

Radon[10] proves the following generalization of Engel form of Cauchy-Schwarz inequality.

Theorem 42 (*Radon's Inequality*). *Let $b_1, \ldots, b_n$ be positive real numbers and $a_1, \ldots, a_n; m$ be non-negative real numbers. Then*

$$\frac{a_1^{m+1}}{b_1^m} + \ldots + \frac{a_n^{m+1}}{b_n^m} \geqslant \frac{(a_1 + \ldots + a_n)^{m+1}}{(b_1 + \ldots + b_n)^m}$$

Beckenbach[11] proves a similar result. For a vector of positive real numbers a and a real number r, let us define

$$\mathfrak{R}_r(\mathrm{a}) = \frac{\sum_{i=1}^{n} a_i^r}{\sum_{i=1}^{n} a_i^{r-1}}$$

Note that this is actually a weighted arithmetic mean of a with the weight vector

$$\omega = \left(\frac{a_1^{r-1}}{\sum_{i=1}^{n} a_i^{r-1}}, \ldots, \frac{a_n^{r-1}}{\sum_{i=1}^{n} a_i^{r-1}}\right)$$

This is also a generalization of arithmetic, geometric and harmonic means. We get the usual arithmetic and harmonic means if we set $r = 1$ and -1 respectively. $r = \frac{1}{2}$ gives us the geometric mean for $n = 2$.

Theorem 43 (*Beckenbach's inequality*). *Let r be a real number. Then*

$$\mathfrak{R}_r(\mathrm{a}+\mathrm{b}) \begin{cases} \leqslant \mathfrak{R}_r(\mathrm{a}) + \mathfrak{R}_r(\mathrm{b}) & \text{if } 1 \leqslant r \leqslant 2 \\ \geqslant \mathfrak{R}_r(\mathrm{a}) + \mathfrak{R}_r(\mathrm{b}) & \text{if } 0 \leqslant r \leqslant 1 \end{cases}$$

Equality occurs if and only if $r = 1$ or a *is proportional to* b.

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[9] Valentin Vornicu
2003 *Olimpiada de matematica: de la provocare la experienta*, GIL Publishing House, vol. v.
[10] Johann Radon
1913 "Über die absolut additiven Mengenfunktionen", *Wien. Sitz. (IIa)*, vol. cxxii, pp. 1295-1438.
[11] Edwin F. Beckenbach
1950 "A Class of Mean Value Functions", *The American Mathematical Monthly*, vol. lvii, no. 1, p. 1, doi: 10.2307/2305163.

Yang[12] proves the following generalization of *Radon's Inequality*.

Theorem 44 (*Generalized Radon's Inequality*). *Let $b_1, \dots, b_n$ be positive real numbers and*

$$a_1, \dots, a_n; r, s$$

be non-negative real numbers such that $r \geqslant s+1$. Then

$$\frac{a_1^r}{b_1^s} + \dots + \frac{a_n^r}{b_n^s} \geqslant \frac{(a_1 + \dots + a_n)^r}{n^{r-s-1}(b_1 + \dots + b_n)^n}$$

Yongtao et al.[13] proves the equivalence between some well known inequalities.

Theorem 45 (*Equivalence of inequalities*). *The following inequalities are mutually equivalent.*

i. *Bernoulli's inequality*

ii. *The weighted arithmetic-geometric mean inequality*

iii. *Hölder's inequality*

iv. *The weighted power mean inequality*

v. *Minkowski's inequality*

vi. *Radon's inequality*

3.4 Smoothing and Isolated Fudging

In many inequalities of the form

$$f(x_1, \dots, x_n) \geqslant a$$

for some real number a, we may be able to establish the property that f assumes smaller values when the difference between two variables x_i and x_j decreases. In such cases, we can use the fact that f assumes the smallest value when $x_1 = \dots = x_n$. This is known as the *smoothing principle*. Recall that we used a similar argument for arithmetic-geometric mean inequality when we replaced the product a_1a_2 by $\bar{a}(\bar{a}+k-h)$ where $a_1 = \bar{a}-h$ and $a_2 = \bar{a}+k$.

[12] Kechang Yang

2002 "A Note and Generalization of a Fractional Inequality", *Journal of Yueyang Normal University (Natural Science Edition)*, vol. XV, no. 4, pp. 9-11, DOI: https://caod.oriprobe.com/articles/18339537/A_Note_and_Generalization_of_a_Fractional_Inequali.htm.

[13] Li Yongtao et al.

2018 "A note on the proofs of generalized Radon inequality", *Mathematica Moravica*, vol. XXII, no. 2, pp. 59-67, DOI: 10.5937/matmor18020591.

Problem 3.4 (USA 1996). Let $a_0, \dots, a_n$ be real numbers in the interval $(0, \frac{\pi}{2})$. If

$$\tan\left(a_0 - \frac{\pi}{4}\right) + \dots + \tan\left(a_n - \frac{\pi}{4}\right) \geqslant n - 1$$

prove that

$$\tan a_0 \cdots \tan a_n \geqslant n^{n+1}$$

Solution. If $x_i = \tan\left(a_i - \frac{\pi}{4}\right)$ so $-1 < x_i < 1$ and

$$\begin{aligned} x_i &= \frac{\tan a_i - 1}{1 + \tan a_i} \\ &= \frac{y_i - 1}{1 + y_i} \end{aligned}$$

where $y_i = \tan a_i$. Then

$$y_i = \frac{1 + x_i}{1 - x_i}$$

3.5 The UVW/PQR/ABC Method

The UVW method was popularized by Rozenberg.[14] It has also been known as the ABC method or the PQR method although it is unclear what exactly the origin of this method is. Although some people believe it was originated at Vietnam where it was called the ABC (Abstract concreteness) method. Currently, Knudsen[15] is the most popular version of this technique but at the core, they all use the same idea. We will try to explain this method as clearly as possible with examples.

Consider an inequality in three variables $a, b, c \in \mathbb{R}$. If the expression is symmetric on a, b, c, then the initial idea of PQR method was to write

$$\begin{aligned} a + b + c &= p \\ ab + bc + ca &= q \\ abc &= r \end{aligned}$$

so that a, b, c are the roots of the equation

$$x^3 - px^2 + qx - r = 0$$

[14] Michael Rozenberg

2011 "uvw–Method in Proving Inequalities", *Math. Ed.*, vol. 59-60, no. 3-4, pp. 6-14, doi: http://www.mathnet.ru/php/archive.phtml?wshow=paper&jrnid=mo&paperid=194&option_lang=eng.

[15] Mathias Tejs Knudsen

n.d. "The UVW-method", *Art of problem solving* (), doi: https://artofproblemsolving.com/community/q2h278791p1507763, https://artofproblemsolving.com/community/q2h278791p1507763.

However, nowadays the most popular way to convert these is to write

$$\begin{aligned} a+b+c &= 3u \\ ab+bc+ca &= 3v^2 \\ abc &= w^3 \end{aligned}$$

Hence, the name UVW. While working with such transformations, be careful not to assume $v^2 \geqslant 0$ by default. We have to consider the case where v^2 is negative as well. We do not know for sure that a, b, c are all positive unless it is stated specifically. However, the following result is very nice when they are indeed positive. We will use the notations as stated above throughout this section and the names of the following theorems follow Knudsen.[16]

Theorem 46 (The Idiot Theorem). *If $a, b, c \geqslant 0$, then $u \geqslant v \geqslant w$.*

Proof. Using $a^2+b^2+c^2 \geqslant ab+bc+ca$ and $a+b+c \geqslant 3\sqrt[3]{abc}$, we immediately see that

$$(a+b+c)^2 \geqslant 3(ab+bc+ca)$$

which implies $9u^2 \geqslant 9v^2$ or $u^2 \geqslant v^2$ and $u \geqslant w$. For proving $v \geqslant w$, we can use arithmetic-harmonic mean inequality

$$\begin{aligned} \frac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}} &\leqslant \sqrt[3]{abc} \\ \iff \frac{3abc}{ab+bc+ca} &\leqslant \sqrt[3]{abc} \\ \iff ab+bc+ca &\geqslant 3\sqrt[3]{(abc)^2} \\ \iff 3v^2 &\geqslant 3w^2 \end{aligned}$$

This proves the theorem. □

Theorem 47 (The UVW theorem). *Let u, v, w be numbers such that u, v^2, w^3 are real numbers. Then there exists real numbers a, b, c such that*

$$\begin{aligned} a+b+c &= 3u \\ ab+bc+ca &= 3v^2 \\ abc &= w^3 \end{aligned}$$

if and only if $u^2 \geqslant v^2$ and

$$3uv^2 - 2u^3 - 2\sqrt{(u^2-v^2)^3} \leqslant w^3 \leqslant 3uv^2 - 2u^3 + 2\sqrt{(u^2-v^2)^3}$$

For proving this theorem, we will need the following result first.

Theorem 48. *a, b, c are real numbers if and only if $(a-b)(b-c)(c-a)$ is a real number.*

[16] Ibid.

Proof. The if part is obvious. So we will focus on the only if part. If $(a-b)(b-c)(c-a)$ is not real, then a,b,c are not real either. Clearly, a,b,c are the roots of $x^3-3ux^2+3vx-w^3=0$ by Vieta's formulas. If one of the roots is complex, say a, then another of b,c is complex, say b. Moreover, we must have $a=z$ and $b=\bar{z}$ for some complex number z since complex roots can appear only in conjugates. Then

$$(a-b)(b-c)(c-a)=(z-\bar{z})(\bar{z}-c)(c-z)$$

Letting $z=u+iv$ for some $u,v\in\mathbb{R}$, we have $\bar{z}=u-iv$ and

$$\begin{aligned}(a-b)(b-c)(c-a)&=-2iv(u-c+iv)(u-c-iv)\\&=-2iv\left((u-c)^2+v^2\right)\end{aligned}$$

This is obviously a complex number, so the claim is true unless $v=0$ which cannot hold since $z\notin\mathbb{R}$. □

Proof of The UVW theorem. For any $x\in\mathbb{R}$, we have $x^2\geqslant 0$. By (48), $a,b,c\in\mathbb{R}$ implies that $(a-b)(b-c)(c-a)\in\mathbb{R}$ and so $(a-b)^2(b-c)^2(c-a)^2\in\mathbb{R}$. Then

$$\begin{aligned}&(a-b)^2(b-c)^2(c-a)^2\geqslant 0\\ \iff\ &27\left(-(w^3-(3uv^2-2u^3))^2+4(u^2-v^2)^3\right)\geqslant 0\\ \iff\ &4(u^2-v^2)^3\geqslant(w^3-(3uv^2-2u^3))^2\\ \iff\ &2\sqrt{(u^2-v^2)^3}\geqslant|w^3-(3uv^2-2u^3)|\end{aligned}$$

This proves the theorem. □

THEOREM 49 (The positivity theorem). *a,b,c are non-negative real numbers if and only if u,v^2,w^3 are non-negative real numbers.*

Proof. The if part is obvious, so we prove the only if part. So we should prove that if any of a,b,c are negative, then at lest one of u,v^2,w^3 is negative. It is easy to see that $w^3=abc$ is negative if one or three of a,b,c are negative. So, we are left with the case where two of a,b,c are negative. Without loss of generality, assume that a and b are negative whereas c is non-negative. Let $a=-x,b=-y$ so that

$$\begin{aligned}a+b+c&=3u\\c-x-y&=3u\\ab+bc+ca&=3v^2\\&=xy-c(x+y)\end{aligned}$$

From this we can show that at least one of u or v has to be negative. Otherwise, if both u and v are non-negative, then $c-x-y\geqslant 0$ and $xy-c(x+y)\geqslant 0$. But this leads to a contradiction since

$$\begin{aligned}xy&\geqslant c(x+y)\\&\geqslant(x+y)^2\\&\geqslant 2xy\end{aligned}$$

by *Arithmetic-Geometric Inequality.* □

THEOREM 50 (Tej's theorem). *Let a,b,c be non-negative real numbers. Then we have the following:*

1. *If at least one value of w^3 exists for fixed u and v^2 corresponding to a,b,c, then w^3 has a global maximum and minimum (see Global maximum and minimum). This maximum value is achieved when at least two of a,b,c are equal. The minimum value is achieved when two of a,b,c are equal or when one of them is 0.*

2. *If at least one value of v^2 exists for fixed u and w^3 corresponding to a, b, c, then v^2 has a global maximum and minimum. This maximum value is achieved when at least two of a, b, c are equal. The minimum value is achieved when two of a, b, c are equal or when one of them is* 0.

3. *If at least one value of u exists for fixed v^2 and w^3 corresponding to a, b, c, then u has a global maximum and minimum. This maximum value is achieved when at least two of a, b, c are equal. The minimum value is achieved when two of a, b, c are equal or when one of them is* 0.

THEOREM 51. *Every symmetric inequality of degree at most* 5 *has to be proved only for $a_1 = a_2$ or $a_n = 0$.*

Proof. Using Elementary polynomials, □

3.6 Mixing Variables Technique

HUNG[17]

3.7 Separation

We prove the arithmetic-geometric inequality yet again. Assume that $x_1 \cdots x_n = 1$ and we want to show that

$$x_1 + \dots + x_n \geqslant n$$

Assume that the result is valid for n and we want to prove it for $n+1$. Without loss of generality, we can assume that $x_1 \leqslant 1$ and $x_2 \geqslant 1$. Otherwise, $x_i \geqslant 1$ for all i implies that $x_i = 1$ which makes the inequality trivially true.

$$\begin{aligned}(1 - x_1)(x_2 - 1) &\leqslant 0 \\ \implies x_1 + x_2 &\geqslant 1 + x_1x_2\end{aligned}$$

Then we have the following by induction.

$$\begin{aligned}x_1 + \dots + x_{n+1} &\geqslant 1 + x_1x_2 + x_3 \cdots x_{n+1} \\ &\geqslant 1 + n\end{aligned}$$

The trick here is to separate the variables in terms of which side of 1 they are on. It is almost the same as being on the same or different sides of a line a point is. See the following problem for a better explanation.

[17] PHAM KIM. HUNG
2007 *Secrets in inequalities*, GIL Publishing House.

Problem 3.5. Let a, b, c be positive real numbers. Prove that

$$a^2 + b^2 + c^2 + 2abc + 1 \geqslant 2(ab + bc + ca)$$

Solution. By the *pigeonhole principle*, at least two of a, b, c are on the same side of 1. Without loss of generality, assume that a and b are on the same side of 1. Then

$$\begin{aligned}(a-1)(b-1) &\geqslant 0\\ \implies ab &\geqslant a + b - 1\end{aligned}$$

Then we have

$$\begin{aligned}a^2 + b^2 + c^2 + 2abc + 1 &\geqslant a^2 + b^2 + c^2 + 2c(a + b - 1) + 1\\ &= (c-1)^2 + a^2 + 2ca + 2bc + b^2\\ &\geqslant 0 + 2ab + 2bc + 2ca\end{aligned}$$

Problem 3.6 (USAMO 2001, problem 3). Let a, b, c be positive real numbers such that

$$a^2 + b^2 + c^2 + 2abc = 4$$

Prove that,

$$0 \leqslant ab + bc + ca - abc \leqslant 2$$

Solution.

Problem 3.7. Let a, b, c be real numbers. Prove that

$$4(1 + a^2)(1 + b^2)(1 + c^2) \geqslant 3(a + b + c)^2$$

Solution. Again, we can assume without loss of generality that $ab + 1 \geqslant a + b$.

$$\begin{aligned}(1 + a^2)(1 + b^2) &= 1 + a^2 + b^2 + a^2b^2\\ &\geqslant 1 + a^2 + b^2 + (a + b - 1)^2\\ &= 2(1 + a^2 + b^2) + 2(ab - a - b)\end{aligned}$$

3.8 Flipping

Sometimes we face inequalities that give us the wrong signs after using some common techniques. We can consider subtracting or dividing some expression to reach the desired form. Let us see how to do that using the following problems.

Problem 3.8. Let $x_1, \dots, x_n$ be real numbers such that $x_1 + \dots + x_n = n$. Prove that,

$$\frac{1}{x_1^2 + 1} + \dots + \frac{1}{x_n^2 + 1} \geqslant \frac{n}{2}$$

Solution. First, see that $x^2+1 \geqslant 2x$ gives us

$$\frac{1}{x_i^2+1} \leqslant \frac{1}{2x_i}$$

This is the opposite sign of what we want to show. One idea to flip the signs is to send the expressions on the other side like this.

$$\begin{aligned}\frac{1}{x_1^2+1}+\ldots+\frac{1}{x_n^2+1}+n &\geqslant \frac{n}{2}+n\\ \iff n &\geqslant \frac{n}{2}+(1-\frac{1}{x_1^2+1})+\ldots+(1-\frac{1}{x_n^2+1})\\ \iff \frac{n}{2} &\geqslant \frac{x_1^2}{x_1^2+1}+\ldots+\frac{x_n^2}{x_n^2+1}\end{aligned}$$

This is evident since

$$\begin{aligned}\frac{1}{x_i^2+1} &\leqslant \frac{1}{2x_i}\\ \iff \frac{x_i^2}{x_i^2+1} &\leqslant \frac{x_i}{2}\end{aligned}$$

Summing over $1 \leqslant i \leqslant n$, we get

$$\begin{aligned}\frac{x_1^2}{x_1^2+1}+\ldots+\frac{x_n^2}{x_n^2+1} &\leqslant \frac{x_1+\ldots+x_n}{2}\\ &= \frac{n}{2}\end{aligned}$$

Problem 3.9 (PuMaC 2014). Let a, b, c be positive real numbers such that $a^2+b^2+c^2=3$. Prove that

$$\frac{1}{a^3+2}+\frac{1}{b^3+2}+\frac{1}{c^3+2} \geqslant 1$$

Solution. We again have a similar situation here. Using *Arithmetic-Geometric Inequality*, we get $a^3+1+1 \geqslant 3a$ so

$$\frac{1}{a^3+2} \leqslant \frac{1}{3a}$$

Let us use the same strategy as the example right above.

$$\begin{aligned}\frac{1}{a^3+2}+\frac{1}{b^3+2}+\frac{1}{c^3+2}+3 &\geqslant 1+3\\ \iff 3 &\geqslant 1+(1-\frac{1}{a^3+2})+(1-\frac{1}{b^3+2})+(1-\frac{1}{c^3+2})\\ \iff 2 &\geqslant \frac{a^3+1}{a^3+2}+\frac{b^3+1}{b^3+2}+\frac{c^3+1}{c^3+2}\end{aligned}$$

We are still facing a problem that the constants in the numerators of each expressions are not vanishing. The reason behind it is the constant 2 which is not fully canceled out. In order to completely cancel it out, we need to subtract the expression from 1/2 instead of 1.

$$\begin{aligned}
\frac{1}{a^3+2}+\frac{1}{b^3+2}+\frac{1}{c^3+2}+\frac{3}{2} &\geqslant 1+\frac{3}{2}\\
\iff \frac{3}{2} &\geqslant 1+(\frac{1}{2}-\frac{1}{a^3+2})+(\frac{1}{2}-\frac{1}{b^3+2})+(\frac{1}{2}-\frac{1}{c^3+2})\\
\iff \frac{1}{2} &\geqslant \frac{a^3}{2(a^3+2)}+\frac{b^3}{2(b^3+2)}+\frac{c^3}{2(c^3+2)}\\
\iff 1 &\geqslant \frac{a^3}{a^3+2}+\frac{b^3}{b^3+2}+\frac{c^3}{c^3+2}
\end{aligned}$$

This is again something we can work on.

$$\begin{aligned}
\frac{1}{a^3+2} &\leqslant \frac{1}{3a}\\
\iff \frac{a^3}{a^3+2} &\leqslant \frac{a^3}{3a}
\end{aligned}$$

Summing over,

$$\frac{a^3}{a^3+2}+\frac{b^3}{b^3+2}+\frac{c^3}{c^3+2}\leqslant\frac{a^2+b^2+c^2}{3}$$

This is exactly what we wanted.

Remark. This can be generalized to the following using the same technique. If $a_1,\ldots,a_n$ are positive real numbers such that $a_1^2+\ldots+a_n^2=n$,

$$\frac{1}{a_1^3+2}+\ldots+\frac{1}{a_n^3+2}\geqslant\frac{n}{3}$$

Problem 3.10 (IMO 2005, problem 3). Let x,y,z be real numbers such that $xyz\geqslant 1$. Prove that

$$\frac{x^5-x^2}{x^5+y^2+z^2}+\frac{y^5-y^2}{x^2+y^5+z^2}+\frac{z^5-z^2}{x^2+y^2+z^5}\geqslant 0$$

Solution. Rearrange the inequality as

$$\begin{aligned}
\frac{x^5-x^2}{x^5+y^2+z^2}+\frac{y^5-y^2}{x^2+y^5+z^2}+\frac{z^5-z^2}{x^2+y^2+z^5} &\geqslant 0\\
\iff \frac{x^5-x^2}{x^5+y^2+z^2}+\frac{y^5-y^2}{x^2+y^5+z^2}+\frac{z^5-z^2}{x^2+y^2+z^5}+3 &\geqslant 3\\
\iff (1-\frac{x^5-x^2}{x^5+y^2+z^2})+(1-\frac{y^5-y^2}{x^2+y^5+z^2})+(1-\frac{z^5-z^2}{x^2+y^2+z^5}) &\leqslant 3\\
\iff \frac{x^2+y^2+z^2}{x^5+y^2+z^2}+\frac{x^2+y^2+z^2}{x^2+y^5+z^2}+\frac{x^2+y^2+z^2}{x^2+y^2+z^5} &\leqslant 3
\end{aligned}$$

Now, there is no obvious way to use something like *Cauchy-Bunyakovsky-Schwarz inequality* or *Engel form of Cauchy-Schwarz* here. And it looks like the denominators are supposed to be in place of the numerators. So, we will try to accomplish that.

$$\begin{aligned} x^5+y^2+z^2 &= \frac{x^4}{\frac{1}{x}}+\frac{y^4}{y^2}+\frac{z^4}{z^2} \\ &\geqslant \frac{(x^2+y^2+z^2)^2}{\frac{1}{x}+y^2+z^2} \\ \Longleftrightarrow \frac{x^2+y^2+z^2}{x^5+y^2+z^2} &\leqslant \frac{\frac{1}{x}+y^2+z^2}{x^2+y^2+z^2} \end{aligned}$$

So we have

$$\begin{aligned} \frac{\frac{1}{x}+y^2+z^2}{x^2+y^2+z^2}+\frac{x^2+\frac{1}{y}+z^2}{x^2+y^2+z^2}+\frac{x^2+y^2+\frac{1}{z}}{x^2+y^2+z^2} &\leqslant \frac{yz+y^2+z^2}{x^2+y^2+z^2}+\frac{x^2+zx+z^2}{x^2+y^2+z^2}+\frac{x^2+y^2+xy}{x^2+y^2+z^2} \\ &= \frac{2(x^2+y^2+z^2)+xy+yz+zx}{x^2+y^2+z^2} \\ &\leqslant \frac{2(x^2+y^2+z^2)+x^2+y^2+z^2}{x^2+y^2+z^2} \end{aligned}$$

This gives us the desired inequality.

Problem 3.11 (APMO 1991, problem 3). Let $a_1, \dots, a_n, b_1, \dots, b_n$ be positive real numbers such that

$$a_1+\dots+a_n=b_1+\dots+b_n$$

Show that

$$\frac{a_1^2}{a_1+b_1}+\dots+\frac{a_n^2}{a_n+b_n} \geqslant \frac{a_1+\dots+a_n}{2}$$

Solution. See the following.

$$\begin{aligned} \frac{a_1^2}{a_1+b_1}+\dots+\frac{a_n^2}{a_n+b_n} &\geqslant \frac{a_1+\dots+a_n}{2} \\ \Longleftrightarrow \frac{a_1^2}{a_1+b_1}+\dots+\frac{a_n^2}{a_n+b_n}+\frac{a_1+\dots+a_n}{2} &\geqslant a_1+\dots+a_n \\ \Longleftrightarrow \frac{a_1+\dots+a_n}{2} &\geqslant (a_1-\frac{a_1^2}{a_1+b_1})+\dots+(a_n-\frac{a_n^2}{a_n+b_n}) \\ &= \frac{a_1b_1}{a_1+b_1}+\dots+\frac{a_nb_n}{a_n+b_n} \\ \Longleftrightarrow a_1+\dots+a_n &\geqslant \frac{2a_1b_1}{a_1+b_1}+\dots+\frac{2a_nb_n}{a_n+b_n} \\ \Longleftrightarrow \frac{1}{2}(a_1+b_1+\dots+a_n+b_n) &\geqslant \frac{2a_1b_1}{a_1+b_1}+\dots+\frac{2a_nb_n}{a_n+b_n} \\ \Longleftrightarrow \frac{a_1+b_1}{2}+\dots+\frac{a_n+b_n}{2} &\geqslant \frac{2a_1b_1}{a_1+b_1}+\dots+\frac{2a_nb_n}{a_n+b_n} \end{aligned}$$

The last inequality is evidently true since $\frac{2ab}{a+b}$ is the harmonic mean of a and b. Then we can use the following to verify the inequality.

$$\frac{a+b}{2} \geqslant \frac{2ab}{a+b}$$

3.9 Neutralizing Denominators

3.10 Dumbassing

3.11 The Equal Variable Method

Cîrtoaje[18]

3.12 Probability in Inequality

[18] Vasile Cîrtoaje
2007 "The Equal Variable Method", *J. Inequal. Pure Appl. Math.*, vol. VIII, no. 1, pp. 1-21, DOI: https://www.emis.de/journals/JIPAM/article828.html#.

CHAPTER 4.

PRACTICE PROBLEMS

We will pose various types of problems here. The problems will not be sorted in any particular order. There will be no categorization and similar problems will not be put together. We did not provide practice problems per section or chapter because in real contest, you will have no way of knowing what technique or theorem to use. This is why we demonstrated some problems while discussing a particular concept but did not add any extra problems. Here, you are free to use any method necessary to solve the problems.

Problem 4.1. Let a, b, c be positive real numbers. Prove that

$$(a^3+2)(b^3+2)(c^3+2) \geqslant (a+b+c)^3$$

Solution. Rearrange the inequality as

$$(a^3+1^3+1^3)^{\frac{1}{3}}(1^3+b^3+1^3)^{\frac{1}{3}}(1^3+1^3+c^3)^{\frac{1}{3}} \geqslant (a^3 1^3 1^3)^{\frac{1}{3}} + (1^3 b^3 1^3)^{\frac{1}{3}} + (1^3 1^3 c^3)^{\frac{1}{3}}$$

which is obviously true by *Generalized Hölder's Inequality*.

Problem 4.2. Let a, b be positive real numbers such that $a+b=1$. Prove that

$$\frac{1}{a^2}+\frac{1}{b^2} \geqslant 8$$

Solution. Using *Generalized Hölder's Inequality* on $(a+b), (a+b), (\frac{1}{a^2}+\frac{1}{b^2})$,

$$(a+b)^{\frac{1}{3}}(a+b)^{\frac{1}{3}}\left(\frac{1}{a^2}+\frac{1}{b^2}\right)^{\frac{1}{3}} \geqslant \left(a\cdot a\cdot\frac{1}{a^2}\right)^{\frac{1}{3}} + \left(b\cdot b\cdot\frac{1}{b^2}\right)^{\frac{1}{3}}$$

Problem 4.3. Let a, b, c be positive real numbers such that $a+b+c=1$. Prove that

$$4a^3+9b^3+36c^3 \geqslant 1$$

Notice that $4, 9, 36$ are square numbers while $a+b+c$ is also 1. This along with the terms like $4a^3$ tells us to somehow cancel the factors $4, 9, 36$ so that we can use the fact that

$$(a^3)^{\frac{1}{3}} + (b^3)^{\frac{1}{3}} + (c^3)^{\frac{1}{3}} = 1$$

Fortunately, we also have $\frac{1}{2}+\frac{1}{3}+\frac{1}{6}=1$ so using *Generalized Hölder's Inequality*,

$$\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{6}\right)^{\frac{1}{3}}\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{6}\right)^{\frac{1}{3}}(4a^3+9b^3+36c^3)^{\frac{1}{3}} \geqslant$$
$$\left(\frac{1}{2}\cdot\frac{1}{2}\cdot 4a^3\right)^{\frac{1}{3}} + \left(\frac{1}{3}\cdot\frac{1}{3}\cdot 9b^3\right)^{\frac{1}{3}} + \left(\frac{1}{6}\cdot\frac{1}{6}\cdot 36c^3\right)^{\frac{1}{3}}$$

This proves the claim.

CHAPTER 5.

CONTEST PROBLEMS

Inequalities have been a part of the mathematical contests for a long time. We have borrowed a lot of problems from these prominent contests. Here is a list of the contests (in no particular order):

- International Mathematical Olympiad (IMO)
- Asian Pacific Mathematical Olympiad (APMO)
- Belarusian National Olympiad (BrNO)
- Bulgarian National Olympiad (BuNO)
- China Mathematical Olympiad (ChMO)
- United States of America Mathematical Olympiad (USAMO)
- Singapore Mathematical Olympiad (SgMO)
- Balkan Mathematical Olympiad (BkMO)
- Abel's Mathematical Contest (Norwegian Mathematical Olympiad) (AMC)
- Austrian-Polish Competition (APC)
- British Mathematical Olympiad (BMO)
- Azerbaijan National Olympiad (AzNO)
- Taiwan National Olympiad (TNO)
- Romania National Olympiad (RNO)
- Albania Mathematical Olympiad (AlMO)
- Poland Mathematical Olympiad (PMO)
- Mediterranean Mathematical Olympiad (MMO)
- Albania-Balkan Mathematical Olympiad (ABMO)
- Brazil National Olympiad (BNO)
- Iranian Mathematical Olympiad (IrMO)

The contests appear in this chapter alphabetically. Problems within a region are ordered based on which year they appeared. In most countries, mathematical Olympiads are followed by team selection tests. However, for better categorization and abbreviation, we have not separated problems based on whether they appeared on a team selection test or national Olympiad. We simply put all the problems under the same region. See Dušan et al.[1] for a reference to the IMO problems (at least up to 2009). See DongPHD and Suugaku[2] for a reference to the APMO problems, although we should warn the reader that the full book is not written in LaTeX for some reason.

Recall that for a vector $\mathrm{a} = (a_1, \dots, a_n)$, the arithmetic, geometric and harmonic means are $\mathfrak{A}(\mathrm{a}), \mathfrak{G}(\mathrm{a}), \mathfrak{H}(\mathrm{a})$ and $\mathfrak{M}_r(\mathrm{a})$ is the power mean of order r. For another vector b, $\langle \mathrm{a}, \mathrm{b} \rangle$ is the inner product of a and b. The L_p norm of a is $\sqrt[p]{a_1^p + \dots + a_n^p}$.

5.1 ABMO

Problem 5.1 (Team Selection Test 2014, problem 1). Prove that for an integer $n > 2$, the following inequality holds:

$$\frac{1}{n+1}\left(1 + \frac{1}{3} + \dots + \frac{1}{2n-1}\right) > \frac{1}{n}\left(\frac{1}{2} + \dots + \frac{1}{2n}\right)$$

Solution.

Problem 5.2 (Team Selection Test 2010, problem 4). Let a, b, c be the sides of a triangle and k be a real number. Prove that

$$a^3 + b^3 + c^3 < k(a + b + c)(ab + bc + ca)$$

holds for $k = 1$. Find the smallest value of k such that the inequality holds.

5.2 AlMO

Problem 5.3 (Albania Team Selection Test 2013, problem 2). Let a, b, c, d be positive real numbers. Prove that $x = 3$ is the minimal value for which the following inequality holds:

$$a^x + b^x + c^x + d^x \geqslant \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}$$

[1] Djukić Dušan et al.
2011 *The IMO compendium: a collection of problems suggested for the International Mathematical Olympiads: 1959-2009*, Springer.

[2] DongPHD and Suugaku
2009 *APMO* 1989 – 2009 *Problems and Solutions*, `vnmath.com`.

Problem 5.4 (Albania Team Selection Test 2012, problem 1). Find the greatest value of the expression

$$\frac{1}{x^2-4x+9}+\frac{1}{y^2-4y+9}+\frac{1}{z^2-4z+9}$$

where x,y,z are non-negative real numbers such that $x+y+z=1$.

5.3 AMC

Problem 5.5 (2014 Norwegian Mathematical Olympiad, problem 1). Let x,y be non-negative real numbers. Show that

$$x^2+y^2+1\leqslant\sqrt{(x^3+y+1)(y^3+x+1)}$$

Problem 5.6 (2013 Norwegian Mathematical Olympiad, problem 1). Find all real numbers a such that

$$3x^2+y^2\geqslant -ax(x+y)$$

holds for all real numbers x,y.

Problem 5.7 (Norwegian Mathematical Olympiad 2012, problem 4). Let x,y be positive real numbers. Show that

$$(1+\frac{x}{y})^3+(1+\frac{y}{x})^3\geqslant 16$$

Problem 5.8 (Norwegian Mathematical Olympiad 2010, problem 2). Let x be a real number such that $0<x<1$. Show that

$$\frac{x^2}{1-x}+\frac{(1-x)^2}{x}\geqslant 1$$

Problem 5.9 (Norwegian Mathematical Olympiad 2009, problem 4). Show that

$$(\frac{2010}{2009})^{2009}>2$$

Problem 5.10 (Norwegian Mathematical Olympiad 2008, problem 3). (i) Let x,y be positive real numbers such that $x+y=2$. Show that

$$\frac{1}{x}+\frac{1}{y}\leqslant\frac{1}{x^2}+\frac{1}{y^2}$$

(ii) Let x,y,z be real numbers such that $x+y+z=2$. Show that

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{4}{9}\leqslant\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}$$

PROBLEM 5.11 (Norwegian Mathematical Olympiad 2006, problem 2). (i) Let a, b be real non-negative numbers. Show that

$$a+b \geqslant \frac{a^2+b^2}{2}+\sqrt{ab}$$

(ii) Let a, b be real numbers in the interval $[0,3]$. Show that

$$\frac{a^2+b^2}{2}+\sqrt{ab} \geqslant \frac{(a+b)^2}{2}$$

PROBLEM 5.12 (Norwegian Mathematical Olympiad 2005, problem 4). (a) Let a, b, c be real positive numbers. Show that

$$(a+b)(a+c) \geqslant 2\sqrt{abc(ab+bc+ca)}$$

(b) Let a, b, c be real numbers such that

$$ab+bc+ca > a+b+c > 0$$

PROBLEM 5.13 (Norwegian Mathematical Olympiad 2003, problem 2). Let $a_1, \dots, a_n$ be n positive integers. Show that

$$\sum_{i=1}^{n} a_i^3 \geqslant \Big(\sum_{i=1}^{n} a_i\Big)^2$$

PROBLEM 5.14 (Norwegian Mathematical Olympiad 2002, problem 2). Let $n \geqslant 2$ be a positive integer and $x_1, \dots, x_n, y_1, \dots, y_n$ be positive real numbers such that

$$x_1+\dots+x_n \geqslant x_1y_1+\dots+x_ny_n$$

Show that

$$x_1+\dots+x_n \leqslant \frac{x_1}{y_1}+\dots+\frac{x_n}{y_n}$$

PROBLEM 5.15 (Norwegian Mathematical Olympiad 2000, problem 2). Let a, b, c, d be non-negative real numbers such that $a+b+c+d=4$. Show that

$$\sqrt{a+b+c}+\sqrt{b+c+d}+\sqrt{c+d+a}+\sqrt{d+a+b} \geqslant 6$$

PROBLEM 5.16 (Norwegian Mathematical Olympiad 1999, problem 1). If a, b, c, d, e are real numbers, prove that

$$a^2+b^2+c^2+d^2+e^2 \geqslant a(b+c+d+e)$$

PROBLEM 5.17 (Norwegian Mathematical Olympiad 1995, problem 4). Let x_i, y_i be positive real numbers for $1 \leqslant i \leqslant n$. Prove that

$$\Big(\sum_{i=1}^{n}(x_i+y_i)^2\Big)\Big(\frac{1}{\sum_{i=1}^{n} x_iy_i}\Big) \geqslant 4n^2$$

Problem 5.18 (Norwegian Mathematical Olympiad 1994, problem 3). Let $x_1, \dots, x_{1994}$ be positive real numbers. Prove that

$$\left(\frac{x_1}{x_2}\right)^{\frac{x_1}{x_2}} \cdots \left(\frac{x_{1993}}{x_{1994}}\right)^{\frac{x_{1994}}{x_{1993}}} \geqslant \left(\frac{x_1}{x_2}\right)^{\frac{x_2}{x_1}} \cdots \left(\frac{x_{1993}}{x_{1994}}\right)^{\frac{x_{1994}}{x_{1993}}}$$

Problem 5.19 (Norwegian Mathematical Olympiad 1993, problem 1). 1. Let a, b, c be sides of a triangle. Prove that

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} < 2$$

Problem 5.20 (Norwegian Mathematical Olympiad 1993, problem 2). Let a, b, c, d are real numbers such that $b < c < d$. Prove that

$$(a+b+c+d)^2 > 8(ac+bd)$$

5.4 APC

Problem 5.21 (APC 2006, problem 5). Prove that for all positive integer n and positive real numbers a, b, c, the following inequality holds:

$$\frac{a^{n+1}}{a^n + a^{n-1}b + \dots + b^n} + \frac{b^{n+1}}{b^n + b^{n-1}c + \dots + c^{n-1}} + \frac{c^{n+1}}{c^n + c^{n-1} + \dots + a^n} \geqslant \frac{a+b+c}{n+1}$$

Problem 5.22 (APC 2005, problem 3). Let $a_0, \dots, a_n$ be real numbers such that

(a) $0 = a_0 \leqslant \dots \leqslant a_n$

(b) For $0 \leqslant i < j \leqslant n$, $a_j - a_i \leqslant j - i$

Prove that

$$\sum_{i=0}^{n} a_i^2 \geqslant \sum_{i=0}^{n} a_i^3$$

Problem 5.23 (APC 2005, problem 10). Determine all pairs of non-negative integers (k, n) such that the following inequality holds for all positive real numbers x, y:

$$1 + \frac{y^n}{x^k} \geqslant \frac{(1+y)^n}{(1+x)^k}$$

Problem 5.24 (APC 2003, problem 8). Given real numbers $x_1 \geqslant \dots \geqslant x_{2003} \geqslant 0$. Show that

$$x_1^n - x_2^n + \dots + x_{2003}^n \geqslant (x_1 - x_2 + \dots + x_{2003})^n$$

for any positive integer n.

Problem 5.25 (APC 2001, problem 3). Let a, b, c be sides of a triangle. Prove that

$$2 \leqslant \frac{a+b}{c} + \frac{b+c}{a} + \frac{c+a}{b} - \frac{a^3+b^3+c^3}{abc} \leqslant 3$$

Problem 5.26 (APC 2000, problem 9). Let a, b, c be non-negative real numbers such that $a+b+c=1$. Prove that

$$2 \leqslant (1-a^2)^2 + (1-b^2)^2 + (1-c^2)^2 \leqslant (1+a)(1+b)(1+c)$$

Problem 5.27 (APC 1999, problem 2). Find the best possible k, l such that

$$k < \frac{v}{v+w} + \frac{w}{w+x} + \frac{x}{x+y} + \frac{y}{y+z} + \frac{z}{z+x} < l$$

for all positive real numbers v, w, x, y, z.

Problem 5.28 (APC 1998, problem 1). Let x_1, x_2, y_1, y_2 be real numbers such that $x_1^2 + x_2^2 \leqslant 1$. Prove that

$$(x_1y_1 + x_2y_2 - 1)^2 \geqslant (x_1^2 + x_2^2 - 1)(y_1^2 + y_2^2 - 1)$$

Problem 5.29 (APC 1997, problem 7). Let p, q be arbitrary real numbers.

(a) Prove that $p^2 + q^2 + 1 > p(q+1)$.

(b) Determine the largest possible b such that $p^2 + q^2 + 1 > bp(q+1)$ for all p, q.

(c) Determine the largest possible c such that $p^2 + q^2 + 1 > cp(q+1)$ for all integer p, q.

Problem 5.30 (APC 1995, problem 9). Prove that for all positive integers m, n and all real numbers x, y, the following inequality holds:

$$(n-1)(m-1)\left(x^{n+m} + y^{n+m}\right) + (n+m-1)\left(x^n y^m + x^m y^n\right) \geqslant nm\left(x^{n+m-1}y + y^{n+m-1}x\right)$$

Problem 5.31 (APC 1993, problem 6). If a, b are non-negative real numbers, prove the inequality

$$\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)^2 \leqslant \frac{a + \sqrt[3]{a^2b} + \sqrt[3]{ab^2} + b}{4} \leqslant \frac{a + \sqrt{ab} + b}{3} \leqslant \sqrt{\left(\frac{a^{2/3} + b^{2/3}}{2}\right)^3}$$

5.5 APMO

Problem 5.32 (2018, problem 2). Let $f(x)$ and $g(x)$ be given by

$$f(x) = \frac{1}{x} + \frac{1}{x-2} + \ldots + \frac{1}{x-2018}$$
$$g(x) = \frac{1}{x-1} + \frac{1}{x-3} + \ldots + \frac{1}{x-2017}$$

Prove that

$$|f(x) - g(x)| > 2$$

PROBLEM 5.33 (2012, problem 5). Let $n \geqslant 2$ be an integer and $a_1, \ldots, a_n$ be real numbers such that $a_1^2 + \ldots + a_n^2 = n$. Prove that

$$\sum_{1 \leqslant i < j \leqslant n} \frac{1}{n - a_i a_j} \leqslant \frac{n}{2}$$

PROBLEM 5.34 (2007, problem 4). Let x, y, z be positive real numbers such that

$$\sqrt{x} + \sqrt{y} + \sqrt{z} = 1$$

Prove that

$$\frac{x^2 + yz}{\sqrt{2x^2(y+z)}} + \frac{y^2 + zx}{\sqrt{2y^2(z+x)}} + \frac{z^2 + xy}{\sqrt{2z^2(x+y)}} \geqslant 1$$

PROBLEM 5.35 (2005, problem 2). Let a, b, c be positive real numbers such that $abc = 8$. Prove that

$$\frac{a^2}{\sqrt{(1+a^3)(1+b^3)}} + \frac{b^2}{\sqrt{(1+b^3)(1+c^3)}} + \frac{c^2}{\sqrt{(1+c^3)(1+a^3)}} \geqslant \frac{4}{3}$$

PROBLEM 5.36 (APMO 2004, problem 5). Prove that the inequality

$$(a^2 + 2)(b^2 + 2)(c^2 + 2) \geqslant 9(ab + bc + ca)$$

holds for all positive real numbers a, b, c.

PROBLEM 5.37 (APMO 2003, problem 4). Let a, b, c be the sides of a triangle with $a + b + c = 1$ and $n \geqslant 2$ be an integer. Show that

$$\sqrt[n]{a^n + b^n} + \sqrt[n]{b^n + c^n} + \sqrt[n]{c^n + a^n} < 1 + \frac{\sqrt[n]{2}}{2}$$

PROBLEM 5.38 (APMO 2002, problem 1). Let n be a positive integer and $a_1, \ldots, a_n$ be a sequence of non-negative integers. Let

$$A_n = \frac{a_1 + \ldots + a_n}{n}$$

Prove that

$$a_1! \cdots a_n! \geqslant (\lfloor A_n \rfloor !)^n$$

PROBLEM 5.39 (APMO 2002, problem 4). Let x, y, z be positive numbers such that

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$$

Show that

$$\sqrt{x + yz} + \sqrt{y + zx} + \sqrt{z + xy} \geqslant \sqrt{xyz} + \sqrt{x} + \sqrt{y} + \sqrt{z}$$

PROBLEM 5.40 (APMO 1998, problem 3). Let a, b, c be positive real numbers. Prove that

$$\left(1 + \frac{a}{b}\right)\left(1 + \frac{b}{c}\right)\left(1 + \frac{c}{a}\right) \geqslant 2\left(1 + \frac{a+b+c}{\sqrt[3]{abc}}\right)$$

Problem 5.41 (APMO 1996, problem 5). Let a, b, c be the lengths of the sides of a triangle. Prove that

$$\sqrt{a+b-c}+\sqrt{b+c-a}+\sqrt{c+a-b} \leqslant \sqrt{a}+\sqrt{b}+\sqrt{c}$$

and determine when equality occurs.

Problem 5.42 (APMO 1991, problem 3). Let $a_1, \ldots, a_n, b_1, \ldots, b_n$ be positive real numbers such that

$$a_1+\ldots+a_n=b_1+\ldots+b_n$$

Show that

$$\frac{a_1^2}{a_1+b_1}+\ldots+\frac{a_n^2}{a_n+b_n} \geqslant \frac{a_1+\ldots+a_n}{2}$$

Solution. Engel form of Cauchy-Schwarz can be used to solve this easily.

$$\begin{aligned}\frac{a_1^2}{a_1+b_1}+\ldots+\frac{a_n^2}{a_n+b_n} &\geqslant \frac{(a_1+\ldots+a_n)^2}{a_1+b_1+\ldots+a_n+b_n}\\ &=\frac{(a_1+\ldots+a_n)^2}{2(a_1+\ldots+a_n)}\\ &=\frac{a_1+\ldots+a_n}{2}\end{aligned}$$

Problem 5.43 (APMO 1990, problem 2). Let $a_1, \ldots, a_n$ be positive real numbers and S_k be the sum of the products of $a_1, \ldots, a_n$ taken k at a time. Show that

$$S_k S_{n-k} \geqslant \binom{n}{k}^2 a_1 \cdots a_n$$

Problem 5.44 (APMO 1989, problem 1). Let $x_1, \ldots, x_n$ be positive real numbers and

$$S=x_1+\ldots+x_n$$

Prove that

$$(1+x_1)\cdots(1+x_n) \leqslant 1+S+\frac{S^2}{2!}+\ldots+\frac{S^n}{n!}$$

5.6 AzNO

Problem 5.45 (2020 National Olympiad, problem 3). a, b, c are positive real numbers such that $a+b+c=3$. Prove that

$$\sum \frac{a^2+6}{2a^2+2b^2+2c^2+2a-1} \leqslant 3$$

Problem 5.46 (2015 National Olympiad, problem 1). Let a, b, c be positive real numbers such that $abc=\frac{1}{8}$. Prove that

$$a^2+b^2+c^2+a^2b^2+b^2c^2+c^2a^2 \geqslant \frac{15}{16}$$

PROBLEM 5.47 (2016 Team Selection Test, problem 1, day 3). Let $a_1, a_2, \ldots$ be a sequence of positive real numbers such that

$$a_{k+1} \geqslant \frac{ka_k}{a_k^2 + k - 1}$$

for every positive integer k. Prove that

$$a_1 + \ldots + a_n \geqslant n$$

for every positive integer n.

PROBLEM 5.48 (2016 Balkan Mathematical Olympiad Team Selection Test 4, problem 1). Let a, b, c be non-negative real numbers. Prove that

$$3(a^2 + b^2 + c^2) \geqslant (a + b + c)\left(\sqrt{ab} + \sqrt{bc} + \sqrt{ca}\right) + (a - b)^2 + (b - c)^2 + (c - a)^2 \geqslant (a + b + c)^2$$

5.7 BkMO

PROBLEM 5.49 (2019, problem 2). Let a, b, c be real numbers such that $0 \leqslant a \leqslant b \leqslant c$ and $a + b + c = ab + bc + ca > 0$. Prove that

$$\sqrt{bc}(a + 1) \geqslant 2$$

Determine when equality occurs.

PROBLEM 5.50 (2015, problem 1). Let a, b, c be positive real numbers. Prove that

$$a^3b^6 + b^3c^6 + c^3a^6 + 3a^3b^3c^3 \geqslant abc(a^3b^3 + b^3c^3 + c^3a^3) + a^2b^2c^2(a^3 + b^3 + c^3)$$

PROBLEM 5.51 (2012, problem 2). Prove that

$$\sum_{cyc}(x + y)\sqrt{(z + x)(z + y)} \geqslant 4(xy + yz + zx)$$

for all positive real numbers x, y, z.

PROBLEM 5.52 (2011, problem 2). Given real numbers x, y, z such that $x + y + z = 0$. Prove that

$$\frac{x(x + 1)}{2x^2 + 1} + \frac{y(y + 2)}{2y^2 + 1} + \frac{z(z + 1)}{2z^2 + 1} \geqslant 0$$

PROBLEM 5.53 (2010, problem 1). Let a, b, c be positive real numbers. Prove that

$$\frac{a^2(b - c)}{a + b} + \frac{b^2(c - a)}{b + c} + \frac{c^2(a - b)}{c + a} \geqslant 0$$

Problem 5.54 (2008, problem 2). Is there a sequence of positive real numbers $a_1, a_2, \ldots$ such that

$$a_1 + \ldots + a_n \leqslant n^2$$
$$\frac{1}{a_1} + \ldots + \frac{1}{a_n} \leqslant 2008$$

for all positive integer n.

Problem 5.55 (2006, problem 1). Let a, b, c be positive real numbers. Prove the inequality

$$\frac{1}{a(b+1)} + \frac{1}{b(c+1)} + \frac{1}{c(a+1)} \frac{3}{1+abc}$$

Problem 5.56 (2005, problem 3). Let a, b, c be positive real numbers. Prove the inequality

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \geqslant a + b + c + \frac{4(a-b)^2}{a+b+c}$$

Problem 5.57 (2001, problem 3). Let a, b, c be positive real numbers such that $a + b + c \leqslant abc$. Prove that

$$a^2 + b^2 + c^2 \geqslant \sqrt{3}abc$$

Problem 5.58 (1998, problem 2). Let $n \geqslant 2$ be an integer and $a_0 < a_1 < \ldots < a_{2n+1}$ be real numbers. Prove that

$$\sqrt[n]{a_1} - \sqrt[n]{a_2} + \ldots + \sqrt[n]{a_{2n+1}} < \sqrt[n]{a_1 - a_2 + \ldots + a_{2n+1}}$$

Problem 5.59 (1993, problem 1). Let a, b, c, d, e, f be real numbers such that $a + b + c + d + e + f = 10$ and

$$(a-1)^2 + (b-1)^2 + (c-1)^2 + (d-1)^2 + (e-1)^2 + (f-1)^2 = 6$$

What is the maximum possible value of f?

5.8 BMO

Problem 5.60 (2005, Round 2, problem 3). Let a, b, c be positive real numbers. Prove that

$$\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right)^2 \geqslant (a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$$

Problem 5.61 (2007, Round 1, problem 5). For positive real numbers a, b, c, prove that

$$(a^2 + b^2 + c^2)^2 \geqslant (a+b+c)(a+b-c)(b+c-a)(c+a-b)$$

Problem 5.62 (2008, Round 1, problem 1). Find the minimum value of $x^2 + y^2 + z^2$ where x, y, z are real numbers such that $x^3 + y^3 + z^3 - 3xyz = 1$.

PROBLEM 5.63 (2010, Round 2, problem 5). For all positive real numbers x, y, z, prove that

$$4(x+y+z)^3 > 27(x^2y+y^2z+z^2x)$$

PROBLEM 5.64 (2011, Round 2, problem 6). Let a, b, c be the sides of a triangle such that $ab+bc+ca=1$. Prove that

$$(a+1)(b+1)(c+1) < 4$$

PROBLEM 5.65 (2015, Round 1, problem 1). Place the following numbers in order and provide your reasoning to do so:

$$3^{3^4}, 3^{4^3}, \ 4^{3^3}, \ 4^{4^3}$$

Here, $a^{b^c} = a^{\left(b^c\right)}$ and not $\left(a^b\right)^c$ which is a^{bc}.

5.9 BNO

PROBLEM 5.66 (2011, problem 6). Let $a_1, \dots, a_{2011}$ be non-negative real numbers such that

$$a_1 + \dots + a_{2011} = \frac{2011}{2}$$

Prove that

$$|(a_1 - a_2)\cdots(a_{2010} - a_{2011})(a_{2011} - a_1)| \leqslant \frac{3\sqrt{3}}{16}$$

PROBLEM 5.67 (2009, problem 3). Let $n > 3$ be a positive integer and $x_1, \dots, x_n$ be positive real numbers. Find the value of

$$\frac{x_1}{x_n + x_1 + x_2} + \frac{x_2}{x_1 + x_2 + x_3} + \dots + \frac{x_n}{x_{n-1} + x_n + x_1}$$

in terms of n.

PROBLEM 5.68 (2008, problem 3). Let x, y, z be real numbers such that $xy+yz+zx = x+y+z$. Find the minimum value of

$$\frac{x}{x^2+1} + \frac{y}{y^2+1} + \frac{z}{z^2+1}$$

PROBLEM 5.69 (2005, problem 2). Determine the smallest possible C such that the inequality

$$C\left(x_1^{2005} + x_3^{2005} + x_4^{2005} + x_5^{2005}\right) \geqslant x_1x_2x_3x_4x_5\left(x_1^{125} + x_2^{125} + x_3^{125} + x_4^{125} + x_5^{125}\right)^{16}$$

PROBLEM 5.70 (2001, problem 1). Show that for any positive real numbers a, b, c,

$$(a+b)(a+c) \geqslant 2\sqrt{abc(a+b+c)}$$

5.10 BrNO

Problem 5.71 (2018, problem 9). Let $n \geqslant 2$ be an integer. Prove the inequality

$$\frac{1}{2!}+\ldots+\frac{2^{n-2}}{n!} \leqslant \frac{3}{2}$$

Problem 5.72 (2015, Day 2, problem 1). Find all real number $x \geqslant -1$ such that the inequality

$$\frac{a_1+x}{2} \cdots \frac{a_n+x}{2} \leqslant \frac{a_1 \cdots a_n+x}{2}$$

holds for all positive integer $n \geqslant 2$ and for all real numbers $a_1, \ldots, a_n \geqslant 1$.

Problem 5.73 (2014 Final Round). Find all values λ such that the inequality

$$\frac{a+b}{2} \geqslant \lambda \sqrt{ab}+(1-\lambda)\sqrt{\frac{a^2+b^2}{2}}$$

Problem 5.74 (2014 Final Round). Prove that for all positive real numbers x and y,

$$\frac{1}{x+y+1}-\frac{1}{(x+1)(y+1)}<\frac{1}{11}$$

Problem 5.75 (2014 Test 2). Let a, b, c be positive real numbers such that

$$ab+bc+ca \geqslant a+b+c$$

Prove that

$$(a+b+c)(ab+bc+ca)+3abc \geqslant 4(ab+bc+ca)$$

Problem 5.76 (2014 Test 4). Let a, b, c be positive real numbers such that $a+b+c=1$. Prove that

$$\frac{a^2}{(b+c)^3}+\frac{b^2}{(c+a)^3}+\frac{c^3}{(a+b)^3} \geqslant \frac{9}{8}$$

Problem 5.77 (2014 Test 6). Let a, b, c be real numbers in the interval $(0,2)$ such that $a+b+c=ab+bc+ca$. Prove that

$$\frac{a^2}{a^2-a+1}+\frac{b^2}{b^2-b+1}+\frac{c^2}{c^2-c+1} \leqslant 3$$

Problem 5.78 (2013 Test 3). Let $n \geqslant 3$ be an integer and $x_1, \ldots, x_n$ be positive real numbers such that $x_1 \cdots x_n=1$. Prove that

$$\frac{x_1^8}{(x_1^4+x_2^4)x_2}+\frac{x_n^8}{(x_n^4+x_1^4)x_1} \geqslant \frac{n}{2}$$

Problem 5.79 (2013 Test 7). Given positive real numbers a, b, c such that

$$\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}=1$$

Prove that

$$\frac{a^2+b^2+c^2+ab+bc+ca-3}{5}\geqslant\frac{a}{b}+\frac{b}{c}+\frac{c}{a}$$

Problem 5.80 (2012 Test 1). Let a, b, c be real numbers such that $0 < a < b < c$. Prove that

$$a^{20}b^{12}+b^{20}c^{12}+c^{20}a^{12} < b^{20}a^{12}+c^{20}b^{12}+b^{20}a^{12}$$

Problem 5.81 (2011 Round 3). Let a, b, x, y be positive real numbers such that

$$ab\geqslant xa+yb$$

Prove that

$$\sqrt{a+b}\geqslant\sqrt{x}+\sqrt{y}$$

Problem 5.82 (2011 Round 3). Let a, b, c, k, l and m be positive real numbers such that

$$abc\geqslant ka+lb+mc$$

Prove that

$$a+b+c\geqslant\sqrt{3}(\sqrt{k}+\sqrt{l}+\sqrt{m})$$

Problem 5.83 (2011 Final Round). Let a, b, c be positive real numbers such that

$$a^2+b^2+c^2=3$$

Prove that

$$a+b+c\geqslant ab+bc+ca$$

Problem 5.84 (2011 Test 5). Let a, b, c be positive real numbers such that

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}=1+\frac{1}{6}(\frac{a}{c}+\frac{c}{b}+\frac{b}{a})$$

Prove that

$$\frac{a^3bc}{b+c}+\frac{b^3ca}{c+a}+\frac{c^3ab}{a+b}\geqslant\frac{1}{6}(ab+bc+ca)^2$$

Problem 5.85 (2010 Final Round). Prove that

$$\frac{a^2}{a+b}+\frac{b^2}{b+c}\geqslant\frac{3a+2b-c}{4}$$

for all positive real numbers a, b and c.

PROBLEM 5.86 (2010 Test 1). Given non-negative real numbers a, b and c such that $a+b+c=1$. Prove that

$$\left(a^2+b^2+c^2\right)^2+6abc \geqslant ab+bc+ca$$

PROBLEM 5.87 (2010 Test 7). Prove that all positive real numbers x, y and z satisfy the inequality

$$x^y+y^z+z^x>1$$

PROBLEM 5.88 (2010 Test 8). Let a, b, c be positive real numbers such that $abc=1$. Prove that

$$\frac{a}{b(a+b)}+\frac{b}{c(b+c)}+\frac{c}{a(c+a)} \geqslant \frac{3}{2}$$

PROBLEM 5.89 (2009 Selection and Training Session). Let a, b and c be positive real numbers. Prove that

$$\frac{1}{(a+b)b}+\frac{1}{(b+c)c}+\frac{1}{(c+a)c} \geqslant \frac{9}{2(ab+bc+ca)}$$

PROBLEM 5.90 (2008, Day 2, problem 1). Let $x_1, \ldots, x_n$ be non-negative real numbers. Prove that

$$\frac{x_1(2x_1-x_2-x_3)}{x_2+x_3}+\ldots+\frac{x_n(2x_n-x_1-x_2)}{x_1+x_2} \geqslant 0$$

PROBLEM 5.91 (2002). For any positive integers a and b, prove that

$$|a\sqrt{2}-b|>\frac{1}{2(a+b)}$$

PROBLEM 5.92 (2002). Given positive real numbers a, b, c and d. Prove that

$$\sqrt{(a+c)^2+(b+d)^2} \leqslant \sqrt{a^2+b^2}+\sqrt{c^2+d^2} \leqslant \sqrt{(a+c)^2+(b+d)^2}+\frac{2|ad-bc|}{\sqrt{(a+c)^2+(b+d)^2}}$$

5.11 BuNO

PROBLEM 5.93 (2020 Day 1, problem 2). Let $b_1, \ldots, b_n$ be non-negative integers and $a_0, a_1, \ldots, a_n$ be real numbers such that $b_1+\ldots+b_n=2$ and $a_0=a_n=0, |a_i-a_{i-1}| \leqslant b_i$ for $1 \leqslant i \leqslant n$. Prove that

$$\sum_{i=1}^{n}(a_i+a_{i-1})b_i \leqslant 2$$

PROBLEM 5.94 (2018, problem 3). Prove that

$$\left(\frac{6}{5}\right)^{\sqrt{3}}>\left(\frac{5}{4}\right)^{\sqrt{2}}$$

Problem 5.95 (2016, problem 3). For positive real numbers a, b, c and d, prove that

$$\frac{a+\sqrt{ab}+\sqrt[3]{abc}+\sqrt[4]{abcd}}{4} \leqslant \sqrt[4]{a\cdot\frac{a+b}{2}\cdot\frac{a+b+c}{3}\cdot\frac{a+b+c+d}{4}}$$

Problem 5.96 (2009, problem 6). Let $a_1, \dots, a_n, b_1, \dots, b_n$ be arbitrarily taken real numbers and $c_1, \dots, c_n$ be positive real numbers, then

$$\left(\sum_{i,j=1}^{n}\frac{a_ia_j}{c_i+c_j}\right)\left(\sum_{i,j=1}^{n}\frac{b_ib_j}{c_i+c_j}\right) \geqslant \left(\sum_{i,j=1}^{n}\frac{a_ib_j}{c_i+c_j}\right)^2$$

Problem 5.97 (2008, problem 3). Let n be a natural number and $a_1, \dots, a_n, b_1, \dots, b_n$ be real positive numbers such that $0 \leqslant a_1 \leqslant \dots \leqslant a_n \leqslant \pi$ and

$$\left|\sum_{i=1}^{n} b_i \cos ka_i\right| < \frac{1}{k}$$

for all positive integer k. Prove that $b_1 = \dots = b_n = 0$.

Problem 5.98 (2007 Team Selection Test, problem 3). Let $n \geqslant 2$ be a positive integer. Find the best constant $C(n)$ such that

$$\sum_{i=1}^{n} x_i \geqslant C(n) \sum_{1\leqslant j<i\leqslant n} (2x_ix_j + \sqrt{x_ix_j})$$

is true for all $x_i \in (0, 1)$ such that

$$(1-x_i)(1-x_j) \geqslant \frac{1}{4}$$

for $1 \leqslant i < j \leqslant n$.

Problem 5.99 (1997, problem 1). Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{1}{1+b+c}+\frac{1}{1+c+b}+\frac{1}{1+a+b} \leqslant \frac{1}{2+a}+\frac{1}{2+b}+\frac{1}{2+c}$$

5.12 ChMO

Problem 5.100 (2019, problem 1). Let $a, b, c, d, e \geqslant -1$ be real numbers such that $a+b+c+d+e = 5$. Find the minimum and maximum value of

$$(a+b)(b+c)(c+d)(d+e)(e+a)$$

Problem 5.101 (2018, problem 6). Let n, k be natural numbers such that $n > k$ and $a_1, \dots, a_n$ be real numbers in the interval $(k-1, k)$. Let $x_1, \dots, x_n$ be positive real numbers such that for any subset I of $\{1, 2, \dots, n\}$ with k elements,

$$\sum_{i\in I} x_i \leqslant \sum_{i\in I} a_i$$

Find the maximum possible value of $x_1 \cdots x_n$.

Problem 5.102 (2017, problem 6). Given an integer $n \geqslant 2$ and real numbers a, b such that $0 < a < b$. Let $x_1, \ldots, x_n$ be real numbers in the interval $[a, b]$. Find the maximum value of

$$\frac{\frac{x_1^2}{x_2} + \ldots + \frac{x_{n-1}^2}{x_n} + \frac{x_n^2}{x_1}}{x_1 + \ldots + x_n}$$

Problem 5.103 (2015, problem 1). Let $z_1, \ldots, z_n$ be complex numbers such that $|z_i - 1| \leqslant r$ for some real number $r \in (0, 1)$. Show that

$$|\sum_{i=1}^{n} z_i| \cdot |\sum_{i=1}^{n} \frac{1}{z_i}| \geqslant n^2(1 - r^2)$$

Problem 5.104 (2011, problem 1). Let $a_1, \ldots, a_n$ be real numbers. Prove that

$$\sum_{i=1}^{n} a_i^2 - \sum_{i=1}^{n} a_i a_{i+1} \leqslant \left\lfloor \frac{n}{2} \right\rfloor (M - m)$$

where $a_{n+1} = a_1, M = \max\{a_1, \ldots, a_n\}, m = \min\{a_1, \ldots, a_n\}$.

Problem 5.105 (2011, problem 5). Let $n \geqslant 4$ be an integer and $a_1, \ldots, a_n, b_1, \ldots, b_n$ be non-negative real numbers such that

$$a_1 + \ldots + a_n = b_1 + \ldots + b_n > 0$$

Find the maximum of

$$\frac{\sum_{i=1}^{n} a_i(a_i + b_i)}{\sum_{i=1}^{n} b_i(a_i + b_i)}$$

Problem 5.106 (2009, problem 4). Let $n > 3$ be an integer and $a_1, \ldots, a_n$ be real numbers satisfying $\min\{a_i - a_j\} \leqslant 1$ for $1 \leqslant i < j \leqslant n$. Find the minimum value of

$$\sum_{i=1}^{n} a_i^3$$

Problem 5.107 (2008, problem 3). Let n be a positive integer and $x_1, \ldots, x_n, y_1, \ldots, y_n$ be real numbers such that

$$x_1 \leqslant x_2 \ldots \leqslant x_n$$
$$y_1 \geqslant y_2 \ldots \geqslant y_n$$
$$\sum_{i=1}^{n} i x_i = \sum_{i=1}^{n} i y_i$$

Show that for any real number α,

$$\sum_{i=1}^{n} x_i \lfloor i\alpha \rfloor \geqslant \sum_{i=1}^{n} y_i \lfloor i\alpha \rfloor$$

PROBLEM 5.108 (2007, problem 1). Let a, b, c be complex numbers and $|a+b| = m, |a-b| = n$. If $mn \neq 0$, show that

$$\max\{|ac+b|, |a+bc|\} \geqslant \frac{mn}{\sqrt{m^2+n^2}}$$

PROBLEM 5.109 (2006, problem 1). Let $a_1, \dots, a_k$ be real numbers such that $a_1 + \dots + a_k = 0$. Prove that

$$\max\{a_1^2, \dots, a_k^2\} \leqslant \frac{k}{3}\left((a_1-a_2)^2 + \dots + (a_{k-1}-a_k)^2\right)$$

PROBLEM 5.110 (2006, problem 5). Let (a_n) be a sequence such that $a_1 = \frac{1}{2}, a_{k+1} = -a_k + \frac{1}{2-a_k}$. Prove that

$$\left(\frac{n}{2(a_1+\dots+a_n)} - 1\right)^n \leqslant \left(\frac{a_1+\dots+a_n}{n}\right)^n \left(\frac{1}{a_1}-1\right)\cdots\left(\frac{1}{a_n}-1\right)$$

PROBLEM 5.111 (2005, problem 1). Let $\theta_i \in (-\frac{\pi}{2}, \frac{\pi}{2})$ for $1 \leqslant i \leqslant 4$. Prove that, there exists a real number x such that

$$\cos^2\theta_1\cos^2\theta_2 - (\sin\theta_1\sin\theta_2 - x)^2 \geqslant 0$$
$$\cos^2\theta_3\cos^2\theta_4 - (\sin\theta_3\sin\theta_4 - x)^2 \geqslant 0$$

if and only if

$$\sum_{i=1}^{4}\sin^2\theta_i \leqslant 2\left(1 + \prod_{i=1}^{4}\sin\theta_i + \prod_{i=1}^{4}\cos\theta_i\right)$$

PROBLEM 5.112 (2004, problem 2). Let $n \geqslant 2$ be an integer and $a_1, \dots, a_n$ are positive integers such that $a_1 < \dots < a_n$ and

$$\frac{1}{a_1} + \dots + \frac{1}{a_n} \leqslant 1$$

Prove that for any real number x,

$$\left(\sum_{i=1}^{n}\frac{1}{a^2+x^2}\right)^2 \leqslant \frac{1}{2}\frac{1}{a_1(a_1-1)+x^2}$$

PROBLEM 5.113 (2003, problem 3). Let n be a positive integer. Find the smallest positive real number λ such that for any $x_1, \dots, x_n \in (0, \frac{\pi}{2})$,

$$\prod_{i=1}^{n}\tan x_i = 2^{\frac{n}{2}}$$

implies

$$\sum_{i=1}^{n}\cos x_i \leqslant \lambda$$

Problem 5.114 (2003, problem 3). Let a, b, c, d be positive real numbers such that $ab + cd = 1$ and $x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4$ are real numbers such that

$$x_i^2 + y_i^2 = 1$$

for $1 \leqslant i \leqslant 4$. Prove that

$$(ax_1 + bx_2 + cx_3 + cx_4)^2 + (ay_4 + by_3 + cy_2 + dy_4)^1 \leqslant 2\left(\frac{a^2 + b^2}{ab} + \frac{c^2 + d^2}{cd}\right)$$

Problem 5.115 (2002, problem 1). For every four points P_1, P_2, P_3, P_4 on the plane, find the minimum value of

$$\frac{\sum_{1 \leqslant i < j \leqslant 4} P_iP_j}{\min\limits_{1 \leqslant i < j \leqslant 4} P_iP_j}$$

Problem 5.116 (2002, problem 3). Let c be a real number such that $c \in (\frac{1}{2}, 1)$. Find the least real number M such that for every integer $n \geqslant 2$ and real numbers $0 \leqslant a_1 \leqslant \ldots \leqslant a_n$, if

$$\frac{1}{n}\sum_{i=1}^{n} ia_i = c\sum_{i=1}^{n} a_i$$

then we always have that

$$\sum_{i=1}^{n} a_i \leqslant M\sum_{i=1}^{m} a_i$$

where $m = \lfloor cn \rfloor$.

Problem 5.117 (1999, problem 5). Determine the maximum value of λ such that

$$f(x) \geqslant \lambda(x - a)^3$$

for all non-negative real number x where

$$f(x) = x^3 + ax^2 + bx + c$$

and has non-negative roots. Find the equality condition.

Problem 5.118 (1998, problem 2). Given a positive integer $n > 1$. Determine with proof if there exists $2n$ distinct positive integers $a_1, \ldots, a_n, b_1, \ldots, b_n$ such that

$$a_1 + \ldots + a_n = b_1 + \ldots + b_n$$
$$n - 1 > \sum_{i=1}^{n} \frac{a_i - b_i}{a_i + b_i} > n - 1 - \frac{1}{1998}$$

Problem 5.119 (1998, problem 6). Let $n \geqslant 2$ be a positive integer and $x_1, \ldots, x_n$ be real numbers such that

$$\sum_{i=1}^{n} x_i^2 + \prod_{i=1}^{n-1} x_ix_{i+1} = 1$$

For each k, find the maximum value of $|x_k|$.

Problem 5.120 (1997, problem 1). Let $x_1, \ldots, x_{1997}$ be real numbers satisfying

$$-\frac{1}{\sqrt{3}} \leqslant x_i \leqslant \frac{1}{\sqrt{3}}$$
$$x_1 + \ldots + x_{1997} = -318\sqrt{3}$$

Determine the maximum value of

$$x_1^{12} + \ldots + x_{1997}^{12}$$

with proof.

Problem 5.121 (1997, problem 6). Let (a_n) be a sequence of real numbers satisfying

$$a_{n+m} \leqslant a_n + a_m$$

for all non-negative integers m, n. Prove that, if $n \geqslant m$,

$$a_n \leqslant ma_1 + (\frac{n}{m} - 1)\, a_m$$

Problem 5.122 (1996, problem 2). Let n be a natural number. Suppose that $x_0 = 0$ and $x_i > 0$ for all $i \in \{1, \ldots, n\}$. If $\sum_{i=1}^{n} x_i = 1$, prove that

$$1 \leqslant \sum_{i=1}^{n} \frac{x_i}{\sqrt{1 + x_0 + \ldots + x_{i-1}}\sqrt{x_i + \ldots + x_n}} < \frac{\pi}{2}$$

Problem 5.123 (1995, problem 1). Let $n \geqslant 3$ be an integer and $a_1, \ldots, a_n, b_1, \ldots, b_n$ be real numbers such that

$$a_1 + \ldots + a_n = b_1 + \ldots + b_n$$
$$0 < a_1 = a_2, a_i + a_{i+1} = a_{i+2}$$
$$0 < b_1 \leqslant b_2, b_i + b_{i+1} \leqslant b_{i+2}$$

Prove that $a_{n-1} + a_n \leqslant b_{n-1} + b_n$.

Problem 5.124 (1995, problem 5). Let $a_1, \ldots, a_{10}$ distinct natural numbers such that $a_1 + \ldots + a_{10} = 1995$. Find the minimum value of

$$a_1a_2 + a_2a_3 + \ldots + a_9a_{10} + a_{10}a_1$$

Problem 5.125 (1993, problem 2). Given a positive integer k and a positive real number a. Find the maximum value of

$$a^{k_1} + \ldots + a^{k_r}$$

where $1 \leqslant r \leqslant k$ and $k_1 + \ldots + k_r = k$.

Problem 5.126 (2019 Team Selection Test, problem 3). Let n be a positive integer and $a_1, \ldots, a_n$ be non-negative real numbers such that

$$a_1 + \ldots + a_n = 1$$

Find the maximum possible value of

$$\sum_{1 \leqslant i < j \leqslant n} \min\{(i-j)^2, (n+i-j)^2 a_i a_j\}$$

Problem 5.127 (2019 Team Selection Test, problem 7). Let x, y, z be complex numbers such that

$$|x|^2 + |y|^2 + |z|^2 = 1$$

Prove that

$$|x^3 + y^3 + z^3 - 3xyz| \leqslant 1$$

Problem 5.128 (2019 Team Selection Test, problem 5). Find all positive integer n such that for any positive real numbers a, b, c, x, y, z the following conditions hold:

$$\begin{aligned} \max\{a, b, c, x, y, z\} &= a \\ a + b + c &= x + y + z \\ abc &= xyz \\ a^n + b^n + c^n &= x^n + y^n + z^n \end{aligned}$$

Problem 5.129 (2018 Team Selection Test, problem 5). Let n, k be positive integers such that $n > 4k$. Find the minimum value of $\lambda(n, k) = \lambda$ such that for any positive real numbers $a_1, \dots, a_n$, we have

$$\sum_{i=1}^{n} \frac{a_i}{\sqrt{a_i^2 + a_{i+1}^2 + \dots + a_{i+k}^2}}$$

where $a_{n+i} = a_i$ for $1 \leqslant i \leqslant k$.

Problem 5.130 (2018 Team Selection Test, problem 3). Let $H(n)$ be the *harmonic sum*

$$1 + \frac{1}{2} + \dots + \frac{1}{n}$$

Prove that there exists a positive constant C such that

$$H(a_1) + \dots + H(a_n) \leqslant C\sqrt{\sum_{i=1}^{n} i a_i}$$

for arbitrary positive integer n and positive real numbers $a_1, \dots, a_n$.

Problem 5.131 (2017 Team Selection Test, problem 2). Let n be positive integers and $x > 1$ be a real number. Prove that

$$\sum_{i=1}^{n} \frac{\{ix\}}{\lfloor ix \rfloor} < \sum_{i=1}^{n} \frac{1}{2i - 1}$$

where $\{a\}$ and $\lfloor a \rfloor$ is the *decimal* and *integer* portion of a respectively.

Problem 5.132 (2017 Team Selection Test, problem 5). Let $m \geqslant 2$ be a positive integer and $a_1, \dots, a_m$ be positive real numbers. Prove that

$$(m-1)^{m-1}\left(x_1^m + \dots + x_m^m\right) \geqslant (x_1 + \dots + x_m)^m - m^m x_1 \cdots x_m$$

Find out when equality holds.

PROBLEM 5.133 (2016 Team Selection Test, problem 2). Find the smallest positive real number λ such that for any complex numbers z_1, z_2, z_3 with $|z_i| < 1$ and $z_1 + z_2 + z_3 = 0$,

$$|z_1z_2 + z_2z_3 + z_3z_1|^2 + |z_1z_2z_3|^2 < \lambda$$

PROBLEM 5.134 (2016 Team Selection Test, problem 7). Let $n > 1$ be an integer and α be a real number such that $0 < \alpha < 2$ and $a_1, \dots, a_n, c_1, \dots, c_n$ be positive real numbers. For a positive real number y, let

$$f(y) = \Big(\sum_{a_i < y} c_i a_i^2\Big)^{\frac{1}{2}} + \Big(\sum_{a_i > y} c_i a_i^{\alpha}\Big)^{\frac{1}{\alpha}}$$

If a positive real number x satisfies $x \geqslant f(y)$ for some positive real number y, prove that $f(x) \leqslant 8^{\frac{1}{\alpha}} x$.

PROBLEM 5.135 (2015 Team Selection Test, problem 4). Let $n \geqslant 2$ be an integer and $x_1, x_2, \dots,$ be a non-decreasing monotonous sequence of positive real numbers such that $x_1, \frac{x_2}{2}, \frac{x_3}{3}, \dots$ is a non-increasing monotonous sequence. Prove that

$$\frac{\sum_{i=1}^n x_i}{n\sqrt[n]{\sum_{i=1}^n x_i}} \leqslant \frac{n+1}{2\sqrt[n]{n!}}$$

PROBLEM 5.136 (2014 Team Selection Test, problem 4). Let (x_n) be a sequence of real numbers and (y_n) be a sequence such that $y_1 = x_1$ and for $n \geqslant 1$,

$$y_{n+1} = x_{n+1} - \sqrt{\sum_{i=1}^n x_i^2}$$

Find the smallest positive real number λ such that for any (x_n) and positive integer m,

$$\frac{1}{m}\sum_{i=1}^n x_i^2 \leqslant \sum_{i=1}^n \lambda^{n-i} y_i^2$$

PROBLEM 5.137 (2014 Team Selection Test, problem 5). Let $n \geqslant 2$ be a positive integer. Find the greatest constant $\lambda(n)$ such that for any non-zero complex numbers $z_1, \dots, z_n$,

$$\sum_{i=1}^n |z_i|^2 \geqslant \lambda(n) \min_{1 \leqslant i \leqslant n} |z_{i+1} - z_i|^2$$

where $z_{n+1} = z_1$.

PROBLEM 5.138 (2013 Team Selection Test, problem 4). Let $n, k > 1$ be integers and $a_1, \dots, a_n$ be non-negative real numbers such that $a_1 \geqslant \dots \geqslant a_n$ and

$$a_1 + \dots + a_n = 1$$
$$c_1 + \dots + c_m \leqslant m^k$$

for any positive integer $m \leqslant n$ and non-negative real numbers $c_1, \dots, c_m$. Find the maximum value of

$$c_1 a_1^k + \dots + c_n a_n^k$$

PROBLEM 5.139 (2013 Team Selection Test, problem 17). Let $n \geqslant 2$ be an integer and $a_1, \dots, a_n, b_1, \dots, b_n$ be non-negative real numbers. Prove that

$$\left(\frac{n}{n-1}\right)^{n-1}\left(\frac{1}{n}\sum_{i=1}^{n} a_i^2\right) + \left(\frac{1}{n}\sum_{i=1}^{n} b_i^2\right) \geqslant \prod_{i=1}^{n} \sqrt[n]{a_i^2 + b_i^2}$$

PROBLEM 5.140 (2012 Team Selection Test, problem 1). Let n be a positive integer and $x_1, \dots, x_n, y_1, \dots, y_n$ be complex numbers such that $|x_i| = |y_i|$ for $1 \leqslant i \leqslant n$. Let

$$x = \frac{1}{n}\sum_{i=1}^{n} x_i$$
$$y = \frac{1}{n}\sum_{i=1}^{n} y_i$$
$$z_i = xy_i - yx_i - x_i y_i$$

Prove that

$$\sum_{i=1}^{n} |z_i| \leqslant n$$

PROBLEM 5.141 (2012 Team Selection Test, problem 4). Let $m, n > 1$ be integers and r, s are real numbers such that $r < s$. Let (a_{ij}) be a $m \times n$ non-zero matrix such that $a_{ij} \geqslant 0$. Find the maximum value of

$$\frac{\left(\sum_{j=1}^{n}\left(\sum_{i=1}^{m} a_{ij}^{s}\right)^{\frac{r}{s}}\right)^{\frac{1}{r}}}{\left(\sum_{i=1}^{m}\left(\sum_{j=1}^{n} a_{ij}^{r}\right)^{\frac{s}{r}}\right)^{\frac{1}{s}}}$$

PROBLEM 5.142 (2011 Team Selection Test, problem 6). Let n be a positive integer. Find the largest real number λ such that for all positive real numbers $x_1, \dots, x_{2n}$ satisfying

$$\frac{1}{2n}\sum_{i=1}^{2n}(x_i + 2)^n \geqslant \prod_{i=1}^{2n} x_i$$

the following inequality is also true:

$$\frac{1}{2n}\sum_{i=1}^{2n}(x_i + 1)^n \geqslant \lambda \prod_{i=1}^{2n} x_i$$

PROBLEM 5.143 (2011 Team Selection Test, problem 7). Let $n \geqslant 3$ be an integer. Find the largest real number M such that for any positive real numbers $x_1, \dots, x_n$ there is an arrangement $y_1, \dots, y_n$ such that

$$\sum_{i=1}^{n} \frac{y_i^2}{y_{i+1}^2 - y_{i+1}y_{i+2} + y_{i+2}^2} \geqslant M$$

where $y_{n+2} = y_2, y_{n+1} = y_1$.

Problem 5.144 (2010 Team Selection Test, problem 1). Let n be a positive integer. The real numbers $a_0, \dots, a_n, b_0, \dots, b_n$ satisfy $a_i + a_{i+1} \geqslant 0$ for $1 \leqslant i \leqslant 2n-1$ and $a_{2i+1} \leqslant 0$ for $1 \leqslant i \leqslant n-1$. For any integers p, q such that $0 \leqslant p \leqslant q \leqslant n$, we have

$$\sum_{i=2p}^{2q} b_i > 0$$

Prove that

$$\sum_{i=0}^{2n} (-1)^i a_i b_i \geqslant 0$$

Determine when equality holds.

Problem 5.145 (2010 Team Selection Test, problem 5). Find all positive real numbers λ such that for all integer $n \geqslant 2$ and all positive real numbers $a_1, \dots, a_n$ such that $a_1 + \dots + a_n = n$,

$$\sum_{i=1}^{n} \frac{1}{a_i} - \lambda \prod_{i=1}^{n} \frac{1}{a_i} \leqslant n - \lambda$$

Problem 5.146 (2010 Team Selection Test, problem 7). Let $n \geqslant 2$ be an integer and a be a positive real number. Find the smallest positive real number $M(n, a) = M$ such that

$$\sum_{i=1}^{n} \frac{1}{a + S - x_i} \geqslant M$$

where $S = \sum_{i=1}^{n} x_i$ for any positive real numbers $x_1, \dots, x_n$.

Problem 5.147 (2010 Team Selection Test 2). Let $n \geqslant 2$ be an integer and $x_1, \dots, x_n$ be real numbers in the interval $[0, 1]$. Prove that there exists real numbers $a_0, \dots, a_n$ such that

$$a_0 + a_n = 0$$
$$|a_i| \leqslant 1$$
$$|a_i - a_{i-1}| = x_i$$

Problem 5.148 (2009 Team Selection Test 2). Let $n \geqslant 2$ be an integer. Find the maximum constant $\lambda(n)$ so that if a sequence of real numbers $a_0, a_1, \dots$ satisfies $0 = a_0 \leqslant a_1 \leqslant \dots \leqslant a_n$ and for $1 \leqslant i \leqslant n-1$, $2a_i \geqslant a_{i-1} + a_{i+1}$ then

$$\Big(\sum_{i=1}^{n} i a_i\Big)^2 \geqslant \lambda(n) \sum_{i=1}^{n} a_i^2$$

Problem 5.149 (2009 Team Selection Test, problem 5). Let $m > 1$ be an integer and n be an odd integer such that $3 \leqslant n < 2m$. Consider a matrix (a_{ij}) such that for any $1 \leqslant j \leqslant n$, $a_{1j}, a_{2j}, \dots, a_{mj}$ is a permutation of $1, 2, \dots, m$ and for any $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n-1$, $|a_{ij} - a_{i(j+1)}| \leqslant 1$. Find the minimum value of

$$\max_{1 < i < m} \sum_{j=1}^{n} a_{ij}$$

Problem 5.150 (2009 Quiz 1, problem 3). Let m, n be positive integers and $x_1, \dots, x_m, y_1, \dots, y_n$ be positive real numbers. Prove that

$$2XY \sum_{i=1}^{m} \sum_{j=1}^{n} |x_i - y_j| \geqslant X^2 \sum_{i=1}^{n} \sum_{j=1}^{n} |y_i - y_j| + Y^2 \sum_{i=1}^{m} \sum_{j=1}^{m} |x_i - x_j|$$

Problem 5.151 (2009 Quiz 5, problem 3). Let a_1, a_2, a_3, a_4 be non-negative real numbers such that $a_1 + a_2 + a_3 + a_4 = 1$. Prove that

$$\max\{\sum_{i=1}^{4} \sqrt{a_i^2 + a_i a_{i-1} + a_{i-1}^2 + a_{i-1} a_{i-2}}, \sum_{i=1}^{4} \sqrt{a_i^2 + a_i a_{i+1} + a_{i+1} a_{i+2}}\} \geqslant 2$$

where $a_{i+4} = a_i$.

Problem 5.152 (2008 Team Selection Test, problem 5). Let $m, n > 1$ be integers and (a_{ij}) be a non-zero matrix of non-negative real numbers. Find the minimum and maximum value of

$$\frac{m \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij}\right)^2 + n \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij}\right)^2}{\left(\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}\right)^2 + mn \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}^2}$$

Problem 5.153 (2008 Team Selection Test, problem 6). Find the maximum constant M such that for any integer $n \geqslant 3$, there exists two sequences of positive real numbers $a_1, \dots, a_n$ and $b_1, \dots, b_n$ satisfying

$$\sum_{i=1}^{n} b_i = 1$$
$$2b_i \geqslant b_{i-1} + b_{i+1}$$
$$a_k^2 \geqslant 1 + \sum_{i=1}^{k} a_i b_i$$

and $a_n = M$.

Problem 5.154 (2008 Quiz 2 problem 3). Let $z_1, z_2, \dots, z_3$ be complex numbers such that $|z_i| \leqslant 1$ and ω_1, ω_2 are the roots of the equation

$$(z - z_1)(z - z_2) + (z - z_2)(z - z_3) + (z - z_3)(z - z_1) = 0$$

Prove that

$$\min\{|z_j - \omega_1|, |z_j - \omega_2\} \leqslant 1$$

Problem 5.155 (2008 Quiz 4 problem 2). Let x, y, z be positive real numbers. Prove that

$$\frac{xy}{z} + \frac{yz}{x} + \frac{zx}{y} > 2\sqrt[3]{x^3 + y^3 + z^3}$$

Problem 5.156 (2008 Quiz 5, problem 2). Let $n \geqslant 2$ be an integer and $a_1, \dots, a_n$ be real numbers not all zero. Determine the necessary and sufficient condition so that there exists a sequence of integers $x_1, \dots, x_n$ which satisfies

$$0 < x_1 < \dots < x_n$$
$$a_1 x_1 + \dots + a_n x_n \geqslant 0$$

Problem 5.157 (2008 Quiz 5, problem 3). Let n be a positive integer and $x_1, \dots, x_n, y_1, \dots, y_n$ be real numbers such that $0 < x_1 \leqslant \frac{x_2}{2} \leqslant \dots \leqslant \frac{x_n}{n}$ and $0 < y_n \leqslant y_{n-1} \leqslant \dots \leqslant y_1$. Prove that

$$\left(\sum_{i=1}^{n} x_i y_i\right)^2 \leqslant \left(\sum_{i=1}^{n} y_i\right)\left(\sum_{i=1}^{n} (x_i^2 - \frac{1}{4}x_i x_{i+1}) y_i\right)$$

where $x_0 = 0$.

Problem 5.158 (2007 Team Selection Test, problem 5). Let $n > 1$ be a positive integer and $x_1, \dots, x_n$ be real numbers satisfying $A = |\sum_{i=1}^{n} x_i| \neq 0$ and $B = \max\limits_{1 \leqslant i \leqslant n} |x_i - x_j| \neq 0$. Prove that for any n vectors $\overrightarrow{\alpha_i}$ in the plane, there exists a permutation $k_1, \dots, k_n$ of the numbers $1, \dots, n$ such that

$$|\sum_{i=1}^{n} a_i x_{k_i} \overrightarrow{\alpha_i}| \geqslant \frac{AB}{2A + B} \max_{1 \leqslant i \leqslant n} |\alpha_i|$$

Problem 5.159 (2007 Quiz 2, problem 1). Let u, v, w be positive real numbers such that

$$u + v + w + \sqrt[3]{uvw} = 4$$

Prove that

$$\sqrt{\frac{uv}{w}} + \sqrt{\frac{vw}{u}} + \sqrt{\frac{wu}{v}} \geqslant u + v + w$$

Problem 5.160 (2007 Quiz 4, problem 1). Let $a_1, \dots, a_n$ be positive real numbers satisfying $a_1 + \dots + a_n = 1$. Prove that

$$(a_1 a_2 + \dots + a_n a_1)\left(\frac{a_1}{a_2^2 + a_2} + \dots + \frac{a_n^2}{a_1^2 + a_1}\right) \geqslant \frac{n}{n+1}$$

Problem 5.161 (2007 Quiz 5, problem 3). Find the smallest constant k such that

$$\frac{x}{\sqrt{x+y}} + \frac{y}{\sqrt{y+z}} + \frac{z}{\sqrt{z+x}} \leqslant k\sqrt{x+y+z}$$

Problem 5.162 (2006 Team Selection Test, problem 3). Let n be a positive integer and $a_1, \dots, a_n$ be real numbers. Prove that there exists real numbers $b_1, \dots, b_n$ such that $a_i - b_i$ is a positive integer for $1 \leqslant i \leqslant n$ and

$$\sum_{1 \leqslant i < j \leqslant n} (b_i - b_j)^2 \leqslant \frac{n^2 - 1}{12}$$

Problem 5.163 (2006 Team Selection Test, problem 8). Let n be a positive integer and $x_1, \dots, x_n$ be positive real numbers such that $x_1 + \dots + x_n = 1$. Prove that

$$\left(\sum_{i=1}^{n} \sqrt{x_i}\right)\left(\sum_{i=1}^{n} \frac{1}{\sqrt{1 + x_i}}\right) \leqslant \frac{n^2}{n+1}$$

Problem 5.164 (2006 Team Selection Test, problem 11). Given positive real numbers x, y, z such that $x + y + z = 1$. Prove that

$$\frac{xy}{\sqrt{xy + yz}} + \frac{yz}{\sqrt{yz + zx}} + \frac{zx}{\sqrt{zx + xy}} \leqslant \frac{\sqrt{2}}{2}$$

PROBLEM 5.165 (2005 Team Selection Test, problem 4). Let $a_1, \dots, a_6; b_1, \dots, b_6; c_1, \dots, c_6$ be permutations of $1, \dots, 6$. Find the minimum value of $\sum_{i=1}^{6} a_i b_i c_i$.

PROBLEM 5.166 (2005 Quiz 1, problem 2). Let a, b, c be non-negative real numbers such that $ab + bc + ca = \frac{1}{3}$. Prove that

$$\frac{1}{a^2 - bc + 1} + \frac{1}{b^2 - ca + 1} + \frac{1}{c^2 - ab + 1} \leqslant 3$$

PROBLEM 5.167 (2005 Quiz 2, problem 3). Let a, b, c, d be positive real numbers such that $abcd = 1$. Prove that

$$\frac{1}{(1+a)^2} + \frac{1}{(1+b)^2} + \frac{1}{(1+c)^2} + \frac{1}{(1+d)^2} \geqslant 1$$

PROBLEM 5.168 (2004 Quiz 4, problem 2). Find the greatest positive real number k such that for any positive real numbers a, b, c, d,

$$(a+b+c)\left(3^4(a+b+c+d)^5 + 2^4(a+b+c+2d)^5\right) \geqslant kabcd^3$$

PROBLEM 5.169 (2003 Quiz 1, problem 1). x, y, z are positive real numbers such that $x + y + z = xyz$. Find the minimum value of

$$x^7(yz-1) + y^7(zx-1) + z^7(xy-1)$$

PROBLEM 5.170 (2003 Quiz 2, problem 3). Let n be a positive integer and the roots of

$$f(z) = z^n + a_1 z^{n-1} + \dots + a_n$$

are $z_1, \dots, z_n$ where $a_1, \dots, a_n$ are complex numbers. If $\sum_{i=1}^{n} |a_i|^2 \leqslant 1$, then prove that $\sum_{i=1}^{n} |z_i|^2 \leqslant n$.

PROBLEM 5.171 (2003 Quiz 6, problem 1). Let n be a positive integer and $a_1, \dots, a_n, x$ be real numbers.

$$g(x) = \sum_{i=1}^{n} a_i \cos ix$$

If $g(x) \geqslant -1$ for all real number x, then prove that $\sum_{i=1}^{n} a_i \leqslant n$.

PROBLEM 5.172 (2003 Quiz 8, problem 3). Let $n \geqslant 2$ be an integer and $a_1, \dots, a_n$ be positive real numbers not all of which are equal such that

$$\sum_{i=1}^{n} \frac{1}{a_i^{2n}} = 1$$

Prove that

$$\left(\sum_{i=1}^{n} a_i^{2n}\right) - n^2 \sum_{1 \leqslant i < j \leqslant n} \left(\frac{a_i}{a_j} - \frac{a_j}{a_i}\right)^2 > n^2$$

PROBLEM 5.173 (2002 Team Selection Test, problem 6). Let

$$f(x_1, x_2, x_3) = -2(x_1^3 + x_2^3 + x_3^3) + 3x_1^2(x_2 + x_3) + 3x_2^2(x_3 + x_1) + 3x_3^2(x_1 + x_2) - 12x_1x_2x_3$$

For real numbers r, s, t, define

$$g(r, s, t) = \max_{t \leqslant x_3 \leqslant t+2} |f(r, r+2, x_3) + s|$$

Find the minimum value of $g(r, s, t)$.

Problem 5.174 (2001 Team Selection Test, problem 4). Let $n > 3$ be an integer. The real numbers $x_1, \dots, x_{n+2}$ satisfy the condition $0 < x_1 < \dots < x_{n+2}$. Find the minimum possible value of

$$\frac{(\sum_{i=1}^{n} \frac{x_{i+1}}{x_i})(\sum_{i=1}^{n} \frac{x_{i+2}}{x_{i+1}})}{(\sum_{i=1}^{n} \frac{x_{i+1}x_{i+2}}{x_{k+1}^2 + x_k x_{k+2}})(\sum_{i=1}^{n} \frac{x_{i+1}^2 + x_i x_{i+2}}{x_i x_{i+1}})}$$

Problem 5.175 (2001 Team Selection Test, problem 6). Find the maximum value of

$$\max_{1 \leqslant x \leqslant 3} |x^3 - ax^2 - bx - c|$$

where a, b, c runs through all real numbers.

Problem 5.176 (1999 Team Selection Test, problem 1). Let n be a positive integer and $x_1, \dots, x_n$ be non-negative real numbers such that $x_1 + \dots + x_n = 1$. Find the largest possible value of $\sum_{i=1}^{n}(x_i^4 - x_j^5)$.

Problem 5.177 (1998 Team Selection Test, problem 3). For a fixed real number $\theta \in [0, \frac{\pi}{2}]$, find the smallest positive real number a for which

$$\frac{\sqrt{a}}{\cos\theta} + \frac{\sqrt{a}}{\sin\theta} > 1$$

and there exists $x \in [1 - \frac{\sqrt{a}}{\sin\theta}, \cos\theta]$ such that

$$((1-x)\sin\theta - \sqrt{a - x^2\cos^2\theta})^2 + (x\cos\theta - \sqrt{a - (1-x)^2\sin\theta})^2 \leqslant a$$

Problem 5.178 (1996 Team Selection Test, problem 5). Let $n \geqslant 4$ be an integer and $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n$ be real numbers such that $\sum_{i=1}^{n} \alpha_i^2 < 1$ and $\sum_{i=1}^{n} \beta_i^2 < 1$.

$$A^2 = 1 - \sum_{i=1}^{n} \alpha_i^2$$
$$B^2 = 1 - \sum_{i=1}^{n} \beta_i^2$$
$$W = \frac{1}{2}(1 - \sum_{i=1}^{n} \alpha_i \beta_i)^2$$

Find all real number λ such that

$$x^n + \lambda(x^{n-1} + \dots + x^3 + Wx^2 + ABx + 1) = 0$$

only has real roots.

Problem 5.179 (1995 Team Selection Test, problem 4). Let n be a positive integer and $r_1, \dots, r_n, s_1, \dots, s_n, t_1, \dots,$

be $5n$ real numbers.

$$R = \frac{1}{n}\sum_{i=1}^{n} r_i$$

$$S = \frac{1}{n}\sum_{i=1}^{n} s_i$$

$$T = \frac{1}{n}\sum_{i=1}^{n} t_i$$

$$U = \frac{1}{n}\sum_{i=1}^{n} u_i$$

$$V = \frac{1}{n}\sum_{i=1}^{n} v_i$$

Prove that

$$\prod_{i=1}^{n} \frac{r_i s_i t_i u_i v_i + 1}{r_i s_i t_i u_i v_i - 1} \geqslant \left(\frac{RSTUV + 1}{RSTUV - 1}\right)^n$$

Problem 5.180 (1993 Team Selection Test, problem 2). Let $n \geqslant 2$ be an integer and a, b, c, d be positive integers such that $\frac{a}{b} + \frac{c}{d} < 1$ and $a + c \leqslant n$. Find the maximum value of $\frac{a}{b} + \frac{c}{d}$ for a fixed n.

5.13 IMO

Problem 5.181 (IMO 1995, problem 2). Let a, b, c be real numbers such that $abc = 1$. Prove that

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geqslant \frac{3}{2}$$

Problem 5.182 (IMO 1999, problem 2). Let $n \geqslant 2$ be an integer. Find the least constant C such that

$$\sum_{1\leqslant i<j\leqslant n} x_i x_j (x_i^2 + x_j^2) \leqslant C\left(\sum_{i=1}^{n} x_i\right)^4$$

holds for all non-negative real numbers $x_1, \dots, x_n$. When does equality occur?

Problem 5.183 (IMO 2000, problem 2). Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\left(a - 1 + \frac{1}{b}\right)\left(b - 1 + \frac{1}{c}\right)\left(c - 1 + \frac{1}{a}\right) \leqslant 1$$

Problem 5.184 (IMO 2001, problem 2). Let a, b, c be positive real numbers. Prove that

$$\frac{a}{\sqrt{a^2 + 8bc}} + \frac{b}{\sqrt{b^2 + 8ca}} + \frac{c}{\sqrt{c^2 + 8ab}} \geqslant 1$$

PROBLEM 5.185 (IMO 2003, problem 5). Let n be a positive integer and $x_1 \leqslant \dots \leqslant x_n$ be real numbers. Prove that

$$\Big(\sum_{i,j=1}^{n} |x_i - x_j|\Big)^2 \leqslant \frac{2(n^2-1)}{3} \sum_{i,j=1}^{n} (x_i - x_j)^2$$

Show that equality holds if and only if $x_1, \dots, x_n$ forms an arithmetic sequence.

PROBLEM 5.186 (IMO 2004, problem 4). Let $n \geqslant 3$ be an integer. Let $t_1, \dots, t_n$ be positive real numbers such that

$$n^2 + 1 > (t_1 + \dots + t_n)\Big(\frac{1}{t_1} + \dots + \frac{1}{t_n}\Big)$$

Show that t_i, t_j, t_k are the sides of a triangle for all $1 \leqslant i < j < k \leqslant n$.

PROBLEM 5.187 (IMO 2005, problem 3). Let x, y, z be real numbers such that $xyz \geqslant 1$. Prove that

$$\frac{x^5 - x^2}{x^5 + y^2 + z^2} + \frac{y^5 - y^2}{x^2 + y^5 + z^2} + \frac{z^5 - z^2}{x^2 + y^2 + z^5} \geqslant 0$$

Solution. We have already solved it in 3.10.

PROBLEM 5.188 (IMO 2006, problem 3). Determine the least real number M such that the inequality

$$|ab(a^2 - b^2) + bc(b^2 - c^2) + ca(c^2 - a^2)| \leqslant M(a^2 + b^2 + c^2)^2$$

holds for all real numbers a, b, c.

PROBLEM 5.189 (IMO 2008, problem 2). Let $x, y, z \neq 1$ be real numbers such that $xyz = 1$. Prove that

$$\frac{x^2}{(x-1)^2} + \frac{y^2}{(y-1)^2} + \frac{z^2}{(z-1)^2} \geqslant 1$$

Also, prove that equality holds for infinitely many rational x, y, z such that $xyz = 1$ and $x, y, z \neq 1$.

PROBLEM 5.190 (IMO 2012, problem 2). Let $n \geqslant 3$ be a positive integer and $a_2, \dots, a_n$ be positive real numbers such that $a_1 \cdots a_n = 1$. Prove that

$$(1 + a_2)^2 \cdots (1 + a_n)^n \geqslant n^n$$

PROBLEM 5.191 (IMO 2020, problem 2). Let a, b, c, d be positive real numbers such that $a \geqslant b \geqslant c \geqslant d$ and $a + b + c + d = 1$. Prove that

$$(a + 2b + 3c + 4d)a^a b^b c^c d^d < 1$$

Solution. Since $a+b+c+d = 1$, by *Weighted Power Mean Inequality* on $\omega = (a, b, c, d)$ and a $= (a, b, c, d)$,

$$a \cdot a + b \cdot a + c \cdot c + d \cdot d \geqslant a^a b^b c^c d^d$$

So it is enough to prove that

$$(a + 2b + 3c + 4d)(a^2 + b^2 + c^2 + d^2) \leqslant (a + b + c + d)^3$$

Expanding these we can easily see that the inequality has to follow. But we can prove it in a smarter way.

$$(a+b+c+d)^3 = (a+b+c+d)(a^2+b^2+c^2+d^2+2\sum ab)$$

where the sum runs over all possible $\binom{4}{2}$ pairs. Then

$$\begin{aligned}(a+2b+3c+4d)(a^2+b^2+c^2+d^2) &< (a+b+c+d)^3\\ \iff (b+2c+3d)(a^2+b^2+c^2+d^2) &< 2(a+b+c+d)(\sum ab)\end{aligned}$$

Now, using $a \geqslant b \geqslant c$,

$$a^2+b^2+c^2+d^2 \leqslant a(a+b+c+d)$$

So, it is enough to show that

$$\begin{aligned}a(b+2c+3d)(a+b+c+d) &\leqslant 2(a+b+c+d)(\sum ab)\\ \iff a(b+2c+3d) &\leqslant 2\sum ab\\ \iff 3da &\leqslant ab+2ca\\ \iff da+da+da &\leqslant ab+ca+ca\end{aligned}$$

This inequality obviously holds.

Remark. We could also use *Weighted Jensen's inequality* after using the fact that $\log(x)$ is concave. The buffalo way works here as well. But the calculation is not going to be pretty if you go that way. This problem was highly criticized within some forums such as the Art of Problem Solving. It was the first inequality problem at the IMO since 2012. A lot of people thought that the days of inequality at the IMO was over. But when this problem appeared at the IMO 2020, many people complained and expressed their disappointment that the *no inequality problem at the IMO* streak was finally broken with such a problem.

Problem 5.192 (IMO Shortlist 2015, A1). Let a,b,c be positive real numbers such that $\min\{ab,bc,ca\} \geqslant 1$. Prove that

$$\sqrt[3]{(a^2+1)(b^2+1)(c^2+1)} \leqslant \left(\frac{a+b+c}{3}\right)^2+1$$

Problem 5.193 (IMO Shortlist 2015, A8). Determine the largest real number a such that for all $n \geqslant 1$ and for all real numbers $x_0,\dots,x_n$ satisfying

$$0 = x_0 < x_1 < \dots < x_n$$

we have

$$\frac{1}{x_1-x_0}+\frac{1}{x_2-x_1}+\dots+\frac{1}{x_n-x_{n-1}} \geqslant a\left(\frac{2}{x_1}+\frac{3}{x_2}+\dots+\frac{n+1}{x_n}\right)$$

Problem 5.194 (IMO Shortlist 2018, A7). Find the maximal value of

$$S = \sqrt[3]{\frac{a}{b+7}}+\sqrt[3]{\frac{c}{d+7}}+\sqrt[3]{\frac{d}{a+7}}$$

where a,b,c,d are non-negative real numbers which satisfy $a+b+c+d=100$.

5.14 IrMO

PROBLEM 5.195 (2002 Team Selection Test, problem 6). Let $x_1, \ldots, x_n$ be positive real numbers such that $x_1^2 + \ldots + x_n^2 = n$, λ be a real number such that $0 \leqslant \lambda \leqslant 1$ and s be a positive real number such that $\sum_{i=1}^{n} x_i \geqslant s$. Prove that at least

$$\left\lceil \frac{s^2(1-\lambda)^2}{n} \right\rceil$$

of these numbers are larger than $\frac{\lambda s}{n}$.

PROBLEM 5.196 (2003 Round 2, problem 1). Let x, y, z be real numbers such that $xyz = -1$. Prove that

$$x^4 + y^4 + z^4 + 3(x+y+z) \geqslant \frac{x^2}{y} + \frac{y^2}{z} + \frac{z^2}{x} + \frac{y^2}{x} + \frac{z^2}{x} + \frac{x^2}{z}$$

PROBLEM 5.197 (2005 Team Selection Test, problem 1). Let $a_1, \ldots, a_n$ be real numbers and

$$\frac{a_1 + \ldots + a_n}{n} = m$$
$$\frac{a_1^2 + \ldots + a_n^2}{n} = 1$$

If there is an i such that $a_i \leqslant m$, prove that

$$n - i \geqslant n(m - a_i)^2$$

PROBLEM 5.198 (2006 Team Selection Test, problem 4). Let $x_1, \ldots, x_n$ be real numbers. Prove that

$$\sum_{i,j=1}^{n} |x_i + x_j| \geqslant n \sum_{i=1}^{n} |x_i|$$

PROBLEM 5.199 (2008 Team Selection Test, problem 5). Let a, b, c be positive real numbers such that $ab + bc + ca = 1$. Prove that

$$\sqrt{a^3 + a} + \sqrt{b^3 + b} + \sqrt{c^3 + c} \geqslant 2\sqrt{a + b + c}$$

PROBLEM 5.200 (2009 Team Selection Test, problem 3). Let a, b, c be positive real numbers such that $a + b + c = 3$. Prove that

$$\frac{1}{2 + a^2 + b^2} + \frac{1}{2 + b^2 + c^2} + \frac{1}{2 + c^2 + a^2} \leqslant \frac{3}{4}$$

PROBLEM 5.201 (2010 Third Round, problem 6). We call a sequence of real numbers $a_{10}, \ldots, a_{1389}$ *concave* if $2a_i \geqslant a_{i-1} + a_{i+1}$ for $0 < i < 1389$. Find the largest real number c such that

$$\sum_{i=0}^{1389} i a_i \geqslant c \sum_{i=0}^{1389} a_i^2$$

for all concave sequence of non-negative real numbers (a_n).

Problem 5.202 (2011 Third Round, problem 10). Find the smallest real number k such that

$$\sqrt{(a^2+1)(b^2+1)(c^2+1)}+\sqrt{(b^2+1)(c^2+1)(d^2+1)}+\sqrt{(c^2+1)(d^2+1)(a^2+1)}+\sqrt{(d^2+1)(a^2+1)(b^2+1)}$$

for all real numbers a, b, c, d.

Problem 5.203 (2012 Third Round, problem 10). Let a, b, c be positive real numbers such that $ab+bc+ca=1$. Show that

$$\sqrt{3}(\sqrt{a}+\sqrt{b}+\sqrt{c}) \leqslant \frac{a\sqrt{a}}{bc}+\frac{b\sqrt{b}}{ca}+\frac{c\sqrt{c}}{ab}$$

Problem 5.204 (2013 Team Selection Test, problem 11). Let a, b, c be the sides of a triangle such that $a \geqslant b \geqslant c$. Prove that

$$\sqrt{a(a+b-\sqrt{ab})}+\sqrt{b(b+c-\sqrt{bc})}+\sqrt{c(c+a-\sqrt{ca})} \geqslant a+b+c$$

Problem 5.205 (2014 Round 2, problem 3). Let x, y, z be non-negative real numbers such that $x^2+y^2+z^2=2(xy+yz+zx)$. Prove that

$$\frac{x+y+z}{3}\sqrt[3]{2xyz}$$

Problem 5.206 (2014 Team Selection Test, problem 5). Let n be a positive integer and $x_1, \dots, x_{n+1}$ be positive real numbers such that $x_1 \cdots x_n = 1$. Prove that

$$\sqrt[x_1]{n}+\dots+\sqrt[x_{n+1}]{n} \geqslant n^{\sqrt[n]{x_1}}+\dots+n^{\sqrt[n]{x_{n+1}}}$$

Problem 5.207 (2014 Team Selection Test 2, problem 5). Let x, y, z be positive real numbers such that

$$x^2+y^2+z^2=x^2y^2+y^2z^2+z^2x^2$$

Prove that

$$\left((x-y)(y-z)(z-x)\right)^2 \leqslant 2\left((x^2-y^2)^2+(y^2-z^2)^2+(z^2-x^2)^2\right)$$

Problem 5.208 (2015 Team Selection Test, problem 6). If a, b, c are positive real numbers such that $a+b+c=abc$, prove that

$$\frac{abc}{3\sqrt{2}}\Big(\sum_{cyc}\frac{\sqrt{a^3+b^3}}{ab+1}\Big) \geqslant \sum_{cyc}\frac{a}{a^2+1}$$

Problem 5.209 (2016 Team Selection Test, problem 1). Let a, b, c, d be positive real numbers such that

$$\frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1}+\frac{1}{d+1}=2$$

Prove that

$$\sqrt{\frac{a^2+1}{2}}+\sqrt{\frac{a^2+1}{2}}+\sqrt{\frac{c^2+1}{2}}+\sqrt{\frac{d^2+1}{2}} \geqslant 3(\sqrt{a}+\sqrt{b}+\sqrt{c}+\sqrt{d})-8$$

PROBLEM 5.210 (2016 Round 2, problem 1). Let a, b, c be positive real numbers such that $c \geqslant b \geqslant a$. Prove that

$$\frac{(c-a)^2}{6c} \leqslant \frac{a+b+c}{3} - \frac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}$$

PROBLEM 5.211 (2017 Round 2, problem 4). Let x, y be two positive real numbers such that $x^4 - y^4 = x - y$. Prove that

$$\frac{x-y}{x^6-y^6} \leqslant \frac{4}{3}(x+y)$$

PROBLEM 5.212 (2017 Team Selection Test, problem 1). Let a, b, c, d be positive real numbers such that $a+b+c+d=2$. Prove that

$$\frac{(a+c)^2}{ad+bc} + \frac{(b+d)^2}{ac+bd} + 4 \geqslant 4\left(\frac{a+b+1}{c+d+1} + \frac{c+d+1}{a+b+1}\right)$$

PROBLEM 5.213 (2018 Team Selection Test, problem 2). Find the smallest real number k such that

$$\left(\frac{2a}{a-b}\right)^2 + \left(\frac{2b}{b-c}\right)^2 + \left(\frac{2c}{c-a}\right)^2 + k4\left(\frac{2a}{a-b} + \frac{2b}{b-c} + \frac{2c}{c-a}\right)$$

for all real numbers a, b, c.

PROBLEM 5.214 (2020 Second Round, problem 2). Let x, y, z be positive real numbers such that $x + y + z = 1399$. Find

$$\max\left(\lfloor x \rfloor y, \lfloor y \rfloor z, \lfloor z \rfloor x\right)$$

5.15 MMO

PROBLEM 5.215 (2018, problem 1). Let $n > 1$ be an integer and $a_1, \ldots, a_n$ be real numbers such that $0 \leqslant a_i \leqslant \frac{\pi}{2}$. Prove that

$$\left(\frac{1}{n}\sum_{i=1}^{n} \frac{1}{1+\sin a_i}\right)\left(1 + \prod_{i=1}^{n} \sqrt[n]{\sin a_i}\right) \leqslant 1$$

PROBLEM 5.216 (2017, problem 4). Let x, y, z and a, b, c be positive real numbers such that $a+b+c=1$. Prove that

$$(x^2+y^2+z^2)\left(\frac{a^3}{x^2+2y^2} + \frac{b^3}{y^2+2z^2} + \frac{c^3}{z^2+2x^2}\right) \geqslant \frac{1}{9}$$

PROBLEM 5.217 (2016, problem 2). Let a, b, c be positive real numbers such that $a+b+c=3$. Prove that

$$\sqrt{\frac{b}{a^2+3}} + \sqrt{\frac{c}{b^2+3}} + \sqrt{\frac{a}{c^2+3}} \leqslant \frac{3}{2}\sqrt[4]{\frac{1}{abc}}$$

Problem 5.218 (2014, problem 1). Let n be a positive integer and $a_1, \dots, a_n, b_1, \dots, b_n$ be $2n$ real numbers. Prove that there exists a positive integer $k \leqslant n$ such that

$$\sum_{i=1}^{n} |a_i - a_k| \leqslant \sum_{i=1}^{n} |b_i - a_k|$$

Problem 5.219 (2013, problem 3). Let x, y, z be positive real numbers such that $x^2y^2 + y^2z^2 + z^2x^2 = 6xyz$. Prove that

$$\sqrt{\frac{x}{x+yz}} + \sqrt{\frac{y}{y+zx}} + \sqrt{\frac{z}{z+xy}} \geqslant \sqrt{3}$$

Problem 5.220 (2010, problem 2). Let $n > 2$ be an integer and $a_1, \dots, a_n$ be positive real numbers such that $a_1 + \dots + a_n = 1$. Prove that

$$\frac{a_2 \cdots a_n}{a_1 + n - 2} + \frac{a_1 a_3 \cdots a_n}{a_2 + n - 2} + \dots + \frac{a_1 \cdots a_{n-1}}{a_n + n - 2} \leqslant \frac{1}{(n-1)^2}$$

Problem 5.221 (2009, problem 4). Let x, y, z be positive real numbers. Prove that

$$\sum_{cyc} \frac{xy}{x^2 + xy + y^2} \leqslant \sum_{cyc} \frac{x}{2x + z}$$

Problem 5.222 (2007, problem 1). Let $x \geqslant y \geqslant z$ be real numbers such that $xy + yz + zx = 1$. Prove that $xz < \frac{1}{2}$. Is it possible to improve the constant $\frac{1}{2}$?

Problem 5.223 (2007, problem 4). Let $x > 1$ be a non-integer real number. Prove that

$$(\frac{x + \{x\}}{\lfloor x \rfloor} - \frac{\lfloor x \rfloor}{x + \{x\}}) + (\frac{x + \lfloor x \rfloor}{\{x\}} - \frac{\{x\}}{x + \lfloor x \rfloor}) > \frac{9}{2}$$

Problem 5.224 (2007, problem 4). Let $(x_{ij})^{m \times n}$ be a matrix of real numbers such that $0 \leqslant x_{ij} \leqslant 1$. Prove that

$$\prod_{j=1}^{n} (1 - \prod_{i=1}^{m} x_{ij}) + \prod_{i=1}^{m} (1 - \prod_{j=1}^{m} (1 - x_{ij})) \geqslant 1$$

Problem 5.225 (2004, problem 3). Let a, b, c be positive real numbers such that $ab + bc + ca + 2abc = 1$. Prove that

$$2(a + b + c) + 1 \geqslant 32abc$$

Problem 5.226 (2003, problem 3). Let a, b, c be non-negative real numbers such that $a + b + c = 3$. Prove that

$$\frac{a}{b^2 + 1} + \frac{b}{c^2 + 1} + \frac{c}{a^2 + 1} \geqslant \frac{3}{2}$$

Problem 5.227 (2002, problem 4). Let a, b, c be non-negative real numbers such that $a^2 + b^2 + c^2 = 1$. Prove that

$$\frac{a}{b^2 + 1} + \frac{b}{c^2 + 1} + \frac{c}{a^2 + 1} \geqslant \frac{3}{4} \left(a\sqrt{a} + b\sqrt{b} + c\sqrt{c}\right)^2$$

Problem 5.228 (1999, problem 3). Let a, b, c be non-zero real numbers and x, y, z be positive real numbers such that $x + y + z = 3$. Prove that

$$\frac{3}{2}\sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}} \geqslant \frac{x}{1 + a^2} + \frac{y}{1 + b^2} + \frac{z}{1 + c^2}$$

5.16 PMO

Problem 5.229 (2019 Finals, problem 5). Let $a_0, a_1, \ldots$ be a sequence of positive real numbers such that a_0 is an integer, $a_i \leqslant a_{i-1} + 1$ for $1 \leqslant i \leqslant n$ and

$$\sum_{i=1}^{n} \frac{1}{a_i} \leqslant 1$$

Prove that

$$n \leqslant 4a_0 \sum_{i=1}^{n} \frac{1}{a_i}$$

Problem 5.230 (2017 Finals, problem 6). Three sequences $a_0, \ldots, a_n; b_0, \ldots, b_n; c_0, \ldots, c_{2n}$ of non-negative real numbers are given such that for all $0 \leqslant i, j \leqslant n$, we have $a_i b_j \leqslant c_{i+j}^2$. Prove that

$$\left(\sum_{i=0}^{n} a_i\right)\left(\sum_{i=0}^{n} b_i\right) \leqslant \left(\sum_{i=0}^{2n} c_i\right)^2$$

Problem 5.231 (2014 Finals, problem 2). Let n be a positive integer, $k \geqslant 2$ be an integer and $a_1, \ldots, a_k; b_1, \ldots, b_n$ be integers such that

$$1 < a_1 < \ldots < a_k < b_1 < \ldots < b_n$$

Prove that if

$$a_1 + \ldots + a_k > b_1 + \ldots + b_n$$

then

$$a_1 \cdots a_k > b_1 \cdots b_n$$

Problem 5.232 (2013 Finals, problem 5). Let k, m, n be distinct positive integers. Prove that

$$\left(k - \frac{1}{k}\right)\left(m - \frac{1}{m}\right)\left(n - \frac{1}{n}\right) \leqslant kmn - (k + m + n)$$

Problem 5.233 (2012 Finals, problem 6). Show that for any positive real numbers a, b, c,

$$\left(\frac{a-b}{c}\right)^2 + \left(\frac{c-a}{b}\right)^2 + \left(\frac{a-b}{c}\right)^2 \geqslant 2\sqrt{2}\left(\frac{a-b}{c} + \frac{b-c}{a} + \frac{c-a}{b}\right)$$

Problem 5.234 (2008 Finals, problem 1). In each cell of an $n \times n$ matrix, a positive integer at most n^2 is written. In the first row, $1, \ldots, n$ are written, in the second $n + 1, \ldots, 2n$ are written and so on. n numbers are selected such that no two numbers are in the same row or column. If a_i is the number chosen from row i, prove that

$$\frac{1}{a_1} + \frac{2^2}{a_2} + \ldots + \frac{n^2}{a_n} \geqslant \frac{n+2}{2} - \frac{1}{n^2 + 1}$$

Problem 5.235 (2004 Finals, problem 4). Let a, b, c be real numbers. Prove that

$$\sqrt{2(a^2+b^2)}+\sqrt{2(b^2+c^2)}+\sqrt{2(c^2+a^2)}\geqslant\sqrt{3(a+b)^2+3(b+c)^2+3(c+a)^2}$$

Problem 5.236 (2003 Finals, problem 2). Let a be a real number such that $0 < a < 1$. Prove that for all finite strictly increasing sequences $k_1, \dots, k_n$ of non-negative integers,

$$\left(\sum_{i=1}^{n} a^{k_i}\right)^2 < \left(\frac{1+a}{1-a}\right)\sum_{i=1}^{n} a^{2k_i}$$

Problem 5.237 (2002 Finals, problem 4). Let $n \geqslant 3$ be an integer and $x_1, \dots, x_n$ be non-negative real numbers. Prove that at least one of the following inequalities are true.

$$\sum_{i=1}^{n}\frac{x_i}{x_{i+1}+x_{i+2}}\geqslant\frac{n}{2}$$

$$\sum_{i=1}^{n}\frac{x_i}{x_{i-1}+x_{i-2}}\geqslant\frac{n}{2}$$

where $x_{n+2}=x_2, x_{n+1}=x_1, x_0=x_n, x_{-1}=x_{n-1}$.

Problem 5.238 (2001 Finals, problem 1). Let n be a positive integer and $x_1, \dots, x_n$ be positive real numbers. Prove that

$$x_1+\dots+nx_n\leqslant\frac{n(n-1)}{2}+x_1+\dots+x_n^n$$

Problem 5.239 (1999 Finals, problem 2). Let n be a positive integer and $a_1, \dots, a_n; b_1, \dots, b_n$ be positive real numbers. Prove that

$$\sum_{1\leqslant i<j\leqslant n}|a_i-a_j|+\sum_{1\leqslant i<j\leqslant n}|b_i-b_j|\leqslant\sum_{i=1}^{n}\sum_{j=1}^{n}|a_i-b_j|$$

Problem 5.240 (1996 Finals, problem 3). Let n be a positive integer and $a_1, \dots, a_n; x_1, \dots, x_n$ be positive real numbers such that

$$a_1+\dots+a_n=x_1+\dots+x_n=1$$

Show that

$$2\sum_{1\leqslant i<j\leqslant n}x_ix_j\leqslant\frac{n-2}{n-1}+\sum_{i=1}^{n}\frac{a_ix_i^2}{1-a_i}$$

When do we have equality?

Problem 5.241 (1995 Finals, problem 1). Let n be a positive integer and $\mathrm{x}=(x_1, \dots, x_n)$ be positive real numbers such that $\mathfrak{H}(\mathrm{x})=1$. Find the smallest possible value of

$$x_1+\dots+\frac{x_n^n}{n}$$

Problem 5.242 (1994 Finals, problem 3). Let $n > 3$ be an integer and $X=\{x_1, \dots, x_n\}$ be set of distinct real numbers such that $\sum_{i=1}^{n} x_i = 0$ and $\sum_{i=1}^{n} x_i^2 = 1$. Show that there are real numbers $a, b, c, d \in X$ such that

$$a+b+c+nabc\leqslant\sum_{i=1}^{n}x_i^3\leqslant a+b+d+nabd$$

5.17 RNO

Problem 5.243 (2019 Team Selection Test, Day 5, problem 1). Determine the largest value of the expression

$$\sum_{1\leqslant i<j\leqslant 4}(x_i+x_j)\sqrt{x_ix_j}$$

where x_1, x_2, x_3, x_4 non-negative real numbers such that $x_1+x_2+x_3+x_4=1$. Also, find the specific values where this maximum is achieved.

Problem 5.244 (2018 Team Selection Test, Day 1, problem 1). Let $x_1, \dots, x_n \geqslant -1$ be real numbers such that $\sum_{i=1}^{n} x_i^3 = 0$. Find the least constant c such that

$$\sum_{i=1}^{n} x_i^2 \leqslant cn$$

Problem 5.245 (2014 Team Selection Test, Day 3, problem 3). Determine the smallest positive constant c such that

$$\sum_{i=1}^{n}\left(\frac{1}{i}\sum_{j=1}^{i}x_j\right)^2 \leqslant c\sum_{i=1}^{n}x_i^2$$

for all positive integer n and all positive real numbers $x_1, \dots, x_n$.

Problem 5.246 (2012 Team Selection Test, Day 2, problem 4). Let k be a positive integer. Find the maximum value of

$$a^{3k-1}b+b^{3k-1}c+c^{3k-1}a+k^2a^kb^kc^k$$

where a, b, c are non-negative real numbers such that $a+b+c=3k$.

Problem 5.247 (2011 Team Selection Test, Day 3, problem 2). Given real numbers x, y, z such that $x+y+z=0$. Show that

$$\frac{x(x+2)}{2x^2+1}+\frac{y(y+2)}{2y^2+1}+\frac{z(z+2)}{2z^2+1}\geqslant 0$$

Problem 5.248 (2011 Team Selection Test, Day 5, problem 2). Let $n \geqslant 2$ be an integer and $x_1, \dots, x_n$ be positive real numbers such that

$$\sum_{i=1}^{n}\frac{1}{x_i+1}=1$$

and $k>1$ be a real number. Show that

$$\sum_{i=1}^{n}\frac{1}{x_i^k+1}\geqslant\frac{n}{(n-1)^k+1}$$

and determine the case of equality.

PROBLEM 5.249 (2010 Team Selection Test 1, problem 2). Let n be a positive integer and $a_1, \dots, a_n$ be positive real numbers. Prove that $f : [0, \infty) \to \mathbb{R}$ defined by

$$f(x) = \frac{a_1 + x}{a_2 + x} + \frac{a_2 + x}{a_3 + x} + \dots + \frac{a_n + x}{a_1 + x}$$

is decreasing.

PROBLEM 5.250 (2010 Team Selection 2, problem 1). Let n be a positive integer. Determine the maximum value of

$$\max\{\frac{x_1}{1 + x_1}, \dots, \frac{x_n}{1 + x_1 + \dots + x_n}\}$$

as $x_1, \dots, x_n$ runs through all non-negative real numbers such that $x_1 + \dots + x_n = 1$.

PROBLEM 5.251 (2010 Team Selection Test 3, problem 1). Let n be a positive integer and $x_1, \dots, x_n$ be positive real numbers such that $x_1 \cdots x_n = 1$. Prove that

$$\sum_{i=1}^{n} x_i^n(1 + x_i) \geqslant \frac{n}{2^{n-1}} \prod_{i=1}^{n}(1 + x_i)$$

PROBLEM 5.252 (2009 Team Selection Test, Day 4, problem 1). Let $n \geqslant 2$ be an integer and $x_1, \dots, x_n$ be positive integers such that $x_1 \leqslant \dots \leqslant x_n$ and $x_1 + \dots + x_n = x_1 \cdots x_n$. What is the maximum possible value of $x_1 + \dots + x_n$.

PROBLEM 5.253 (2008 Team Selection Test, Day 1, problem 2). Let $n \geqslant 2$ be an integer and $a_1, \dots, a_n; b_1, \dots, b_n$ be positive real numbers such that $a_i < b_i$ and

$$b_1 + \dots + b_n < 1 + a_1 + \dots + a_n$$

Prove that there exists a real number c such that

$$(a_i + c + k)(b_i + c + k) > 0$$

for $1 \leqslant i \leqslant n$ and any integer k.

PROBLEM 5.254 (2008 Team Selection Test, Day 2, problem 1). Let $n \geqslant 3$ be an odd integer. Determine the maximum value of

$$\sqrt{|x_1 - x_2|} + \dots + \sqrt{|x_n - x_1|}$$

where $x_1, \dots, x_n$ are real numbers in the interval $[0, 1]$.

PROBLEM 5.255 (2007 Team Selection Test, Day 1, problem 1). Let $a_1, \dots, a_n$ be non-negative real numbers such that $a_1^2 + \dots + a_n^2 = 1$. Find the maximum value of

$$(1 - a_1) \cdots (1 - a_n)$$

PROBLEM 5.256 (2007 Team Selection Test, Day 3, problem 2). Let $n, p \geqslant 4$ be integers and $x_1, \dots, x_n$ be positive real numbers such that $x_1 + \dots + x_n = n$. Prove that

$$\sum_{i=1}^{n} \frac{1}{x_i^p} \geqslant \sum_{i=1}^{n} x_i^p$$

is false.

Problem 5.257 (2007 Team Selection Test, Day 7, problem 1). Let $n \geqslant 2$ be an integer and $a_1, \dots, a_n; b_1, \dots, b_n$ be real numbers such that

$$\sum_{i=1}^{n} a_i^2 = \sum_{i=1}^{n} b_i^2 = 1$$

$$\sum_{i=1}^{n} a_i b_i = 0$$

Prove that

$$\left(\sum_{i=1}^{n} a_i\right)^2 + \left(\sum_{i=1}^{n} b_i\right)^2 \leqslant n$$

Problem 5.258 (2006 Team Selection Test, Day 1, problem 4). Let n be a positive integer and $a_1, \dots, a_n$ be real numbers such that $|a_i| \leqslant 1$ for $1 \leqslant i \leqslant n$ and $a_1 + \dots + a_n = 0$. Prove that there exists a positive integer $k \leqslant n$ such that

$$|1 \cdot a_1 + \dots + k a_k| \leqslant \frac{2k+1}{4}$$

Also show that this is the best bound possible for $n > 2$.

Problem 5.259 (2006 Team Selection Test, Day 2, problem 4). Let n be a positive integer and $x_1, \dots, x_n$ be real numbers. Prove that

$$\sum_{1 \leqslant i < j \leqslant n} |x_i + x_j| \geqslant \frac{n-2}{2} \sum_{i=1}^{n} |x_i|$$

Problem 5.260 (2005 Team Selection Test, Day 5, problem 2). Let $n \geqslant 2$ be an integer and $x_1, \dots, x_n$ be positive real numbers such that $x_1 \cdots x_n = 1$. Find the smallest real value $\varrho(n)$ such that for any $x_1, \dots, x_n$,

$$\sum_{i=1}^{n} \frac{1}{x_i} \leqslant \sum_{i=1}^{n} x_i^r$$

is true for all $r \geqslant \varrho(n)$.

Problem 5.261 (2004 Team Selection Test, Day 1, problem 1). Let $a_1, \dots, a_4$ be the sides of a quadrilateral with perimeter $2s$. Prove that

$$\sum_{i=1}^{4} \frac{1}{a_i + s} \leqslant \frac{2}{9} \sum_{1 \leqslant i < j \leqslant 4} \frac{1}{\sqrt{(s-a_i)(s-a_j)}}$$

When does equality hold?

Problem 5.262 (2002 Team Selection Test, Day 3, problem 2). Let $n \geqslant 4$ be an integer and $a_1, \dots, a_n$ be positive real numbers such that

$$a_1^2 + \dots + a_n^2 = 1$$

Prove that

$$\frac{a_1}{a_1^2+1} + \dots + \frac{a_n}{a_n^2+1} \geqslant \frac{4}{5}\left(a_1\sqrt{a_1} + \dots + a_n\sqrt{a_n}\right)^2$$

PROBLEM 5.263 (2001 Team Selection Test, Day 1, problem 3). Let a, b, c be the sides of a triangle. Prove that

$$(-a+b+c)(a-b+c)+(a-b+c)(a+b-c)+(a+b-c)(-a+b+c) \leqslant \sqrt{abc}(\sqrt{a}+\sqrt{b}+\sqrt{c})$$

PROBLEM 5.264 (2000 Team Selection Test, Day 1, problem 2). Let n be a positive integer and $x_1, \ldots, x_n$ be real numbers such that $|x_{k+1}-x_k| \leqslant 1$ for $1 \leqslant k \leqslant n-1$. Prove that

$$\sum_{i=1}^{n}|x_i| - |\sum_{i=1}^{n} x_i| \leqslant \frac{n^2-1}{4}$$

PROBLEM 5.265 (1999 Team Selection Test, Day 2, problem 2). Let n and $x_1, \ldots, x_n$ be positive integers. Prove that

$$x_1^2+\ldots+x_n^2 \geqslant \frac{2n+1}{3}(x_1+\ldots+x_n)$$

PROBLEM 5.266 (1996 Team Selection Test, Day 2, problem 4). If $p_1, \ldots, p_k$ are the distinct prime divisors of n, define

$$a_n = \frac{1}{p_1}+\ldots+\frac{1}{p_n}$$

Prove that

$$\sum_{i=2}^{n} a_1 \cdots a_i < 1$$

for $n \geqslant 2$.

PROBLEM 5.267 (1996 Team Selection Test, Day 3, problem 1). Let $n \geqslant 3$ be an integer and $x_1, \ldots, x_{n-1}$ be non-negative integers such that

$$x_1+\ldots+x_{n-1} = n$$
$$1\cdot x_1+\ldots+(n-1)x_n = 2(n-2)$$

Find the minimum value of

$$\sum_{i=1}^{n} i(2n-i)x_i$$

PROBLEM 5.268 (1996 Team Selection Test, Day 4, problem 1). Let n be a positive integer and $x_1, \ldots, x_n$ be positive real numbers such that

$$x_{n+1} = x_1+\ldots+x_n$$

Prove that

$$\sum_{i=1}^{n}\sqrt{x_i(x_{n+1}-x_i)} \leqslant \sqrt{\sum_{i=1}^{n} x_{n+1}(x_{n+1}-x_i)}$$

PROBLEM 5.269 (1993 Team Selection Test, Day 1, problem 1). Find the maximum possible constant A such that

$$\frac{x}{\sqrt{y^2+z^2}}+\frac{y}{\sqrt{z^2+x^2}}+\frac{z}{\sqrt{x^2+y^2}} \geqslant A$$

for all positive real numbers x, y, z.

5.18 SgMO

Problem 5.270 (Singapore Team Selection Test 2009, problem 2). If a, b, c are three positive real numbers such that $ab + bc + ca = 1$, prove that

$$\sqrt[3]{\frac{1}{a} + 6b} + \sqrt[3]{\frac{1}{b} + 6c} + \sqrt[3]{\frac{1}{c} + 6a} \leqslant \frac{1}{abc}$$

Problem 5.271 (Singapore Team Selection Test 2008, problem 2). Let $x_1, \dots, x_n$ be positive real numbers such that $x_1 \cdots x_n = 1$. Prove that

$$\sum_{i=1}^{n} \frac{1}{n - 1 + x_i} \leqslant 1$$

Problem 5.272 (Singapore Team Selection Test 2007, problem 2). Prove the inequality

$$\sum_{i<j} \frac{a_i a_j}{a_i + a_j} \leqslant \frac{n}{2(a_1 + \dots + a_n)} \sum_{i<j} a_i a_j$$

for all positive real numbers $a_1, \dots, a_n$.

Problem 5.273 (Singapore 2006, problem 2). Let $n > 1$ be an integer and $x_1, \dots, x_n$ be real numbers such that

$$|x_1| + \dots + |x_n| = 1 \quad \text{and}$$
$$x_1 + \dots + x_n = 0$$

Prove that

$$\left|\frac{x_1}{1} + \dots + \frac{x_n}{n}\right| \leqslant \frac{1}{2}\left(1 - \frac{1}{n}\right)$$

Problem 5.274 (Singapore 2004, problem 2). Let a, b, c be real numbers such that $ab + bc + ca = 1$ and $0 < a, b, c < 1$. Prove that

$$\frac{a}{1 - a^2} + \frac{b}{1 - b^2} + \frac{c}{1 - c^2} \geqslant \frac{3\sqrt{3}}{2}$$

5.19 TNO

Problem 5.275 (2005 Day 1, problem 1). Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$1 + \frac{3}{a + b + c} \geqslant \frac{6}{ab + bc + ca}$$

Problem 5.276 (2005 Day 2, problem 3). Let $a_1, \dots, a_{95}$ be positive real numbers. Prove that

$$\sum_{i=1}^{95} a_i \leqslant 94 + \prod_{i=1}^{n} \max\{1, a_i\}$$

Problem 5.277 (2002 Day 1, problem 3). Let $x, y, z, a, b, c, d, e, f$ be real numbers satisfying

$$\begin{aligned}\max\{a, 0\} + \max\{b, 0\} &< x + ay + bz < 1 + \min\{a, 0\} + \min\{b, 0\}\\ \max\{c, 0\} + \max\{d, 0\} &< cx + y + dz < 1 + \min\{c, 0\} + \min\{d, 0\}\\ \max\{e, 0\} + \max\{f, 0\} &< ex + fy + z < 1 + \min\{e, 0\} + \min\{d, 0\}\end{aligned}$$

Prove that $0 < x, y, z < 1$.

Problem 5.278 (2002 Day 2, problem 1). Let $x_1, , x_2, x_3, x_4$ be positive real numbers at most $\frac{1}{2}$. Prove that

$$\frac{x_1x_2x_3x_4}{(1-x_1)(1-x_2)(1-x_3)(1-x_4)} \leqslant \frac{x_1^4 + x_2^4 + x_3^4 + x_4^4}{(1-x_1)^4 + (1-x_2)^4 + (1-x_3)^4 + (1-x_4)^4}$$

Problem 5.279 (1996 Day 1, problem 2). Let a be a positive real number at most 1 and $a_1, \dots, a_{1996}$ be real numbers such that $a \leqslant a_i \leqslant \frac{1}{a_i}$. If $k_1, \dots, k_{1996}$ are any non-negative real numbers such that $k_1 + \dots + k_{1996} = 1$, prove that

$$\left(\sum_{i=1}^{1996} k_ia_i\right)\left(\sum_{i=1}^{n} \frac{k_i}{a_i}\right) \leqslant \left(a + \frac{1}{a}\right)^2$$

Problem 5.280 (1996 Day 2, problem 2). Determine integers $a_1, \dots, a_{99} = a_0$ satisfying $|a_k - a_{k+1}| \geqslant 1996$ for $1 \leqslant k \leqslant 1995$ such that

$$\max_{1 \leqslant i \leqslant 1995} |a_i - a_{i+1}|$$

is as minimum as possible. What is this minimum value?

Problem 5.281 (1994 Day 1, problem 2). Let a, b, c be positive real numbers and α be a real number.

$$\begin{aligned}f(\alpha) &= abc\left(a^\alpha + b^\alpha + c^\alpha\right)\\ g(\alpha) &= a^{2+\alpha}(b + c - a) + b^{2+\alpha}(c + a - b) + c^{2+\alpha}(a + b - c)\end{aligned}$$

Determine $\min |f(\alpha) - g(\alpha)|$ and $\max |f(\alpha) - g(\alpha)$ if they exist.

5.20 USAMO

Problem 5.282 (USAMO 2020, problem 6). Let $n \geqslant 2$ be an integer and $x_1 \geqslant \dots \geqslant x_n, y_1 \geqslant \dots \geqslant y_n$ be $2n$ real numbers such that

$$\begin{aligned}0 &= x_1 + \dots + x_n = y_1 + \dots + y_n\\ 1 &= x_1^2 + \dots + x_n^2 = y_1^2 + \dots + y_n^2\end{aligned}$$

Prove that

$$\sum_{i=1}^{n}(x_iy_i - x_iy_{n+i-1})^2 \geqslant \frac{2}{\sqrt{n-1}}$$

Problem 5.283 (USAMO 2018, problem 1). Let a, b, c be positive real numbers such that $a + b + c = 4\sqrt[3]{abc}$. Prove that

$$2(ab + bc + ca) + 4\min\{a^2, b^2, c^2\} \geqslant a^2 + b^2 + c^2$$

Problem 5.284 (USAMO 2017, problem 6). Find the minimum possible value of

$$\frac{a}{b^3+4} + \frac{b}{c^3+4} + \frac{c}{d^3+4} + \frac{d}{a^3+4}$$

given that a, b, c, d are non-negative real numbers such that $a + b + c + d = 4$.

Problem 5.285 (USAMO 2013, problem 4). Find all real numbers $x, y, z \geqslant 1$ such that

$$\min\{\sqrt{x+xyz}, \sqrt{y+xyz}, \sqrt{z+xyz}\} = \sqrt{x-1} + \sqrt{y-1} + \sqrt{z-1}$$

This is actually a special case of the following.

Problem 5.286. Prove that for real numbers $a, b, c \geqslant 1$, the following inequality holds:

$$\sqrt{a-1} + \sqrt{b-1} + \sqrt{c-1} \leqslant \sqrt{a(bc+1)}$$

Problem 5.287 (USAMO 2012, problem 6). Let $n \geqslant 2$ be an integer and $x_1 \geqslant \ldots \geqslant x_n, y_1 \geqslant \ldots \geqslant y_n$ be $2n$ real numbers such that

$$0 = x_1 + \ldots + x_n = y_1 + \ldots + y_n$$
$$1 = x_1^2 + \ldots + x_n^2 = y_1^2 + \ldots + y_n^2$$

For each subset A of $\{1, 2, \ldots, n\}$, define

$$S_A = \sum_{i \in A} x_i$$

$S_A = 0$ if A is empty. Prove that for any positive number λ, the number of sets A satisfying $S_A > \lambda$ is at most

$$\frac{2^{n-3}}{\lambda^2}$$

For which choices of $x_1, \ldots, x_n, \lambda$ does equality hold?

Problem 5.288 (USAMO 2011, problem 1). Let a, b, c be positive real numbers such that

$$a^2 + b^2 + c^2 + (a+b+c)^2 \leqslant 4$$

Prove that

$$\frac{ab+1}{(a+b)^2} + \frac{bc+1}{(b+c)^2} + \frac{ca+1}{(c+a)^2} \geqslant 3$$

Problem 5.289 (USAMO 2009, problem 4). Let $n \geqslant 2$ be an $a_1, \ldots, a_n$ be positive real numbers such that

$$(a_1 + \ldots + a_n)(\frac{1}{a_1} + \ldots + \frac{1}{a_n}) \leqslant (n + \frac{1}{2})^2$$

Prove that

$$\max\{a_1, \ldots, a_n\} \leqslant 4 \min\{a_1, \ldots, a_n\}$$

Problem 5.290 (USAMO 2004, problem 5). Let a, b, c be positive real numbers. Prove that

$$(a^5 - a^2 + 3)(b^5 - b^2 + 3)(c^5 - c^2 + 3) \geqslant (a + b + c)^3$$

Problem 5.291 (USAMO 2003, problem 5). Let a, b, c be positive real numbers. Prove that

$$\frac{(2a + b + c)^2}{2a^2 + (b + c)^2} + \frac{(2b + c + a)^2}{2b^2 + (c + a)^2} + \frac{(2c + a + b)^2}{2c^2 + (a + b)^2} \leqslant 8$$

Solution. We have already solved it in 3.3.

Problem 5.292 (USAMO 2001, problem 3). Let a, b, c be positive real numbers such that $a^2 + b^2 + c^2 + abc = 4$. Prove that

$$0 \leqslant ab + bc + ca - abc \leqslant 2$$

Problem 5.293 (USAMO 2000, problem 6). Let $a_1, \ldots, a_n, b_1, \ldots, b_n$ be non-negative real numbers. Prove that

$$\sum_{i,j=1}^{n} \min\{a_i a_j, b_i b_j\} \leqslant \sum_{i,j=1}^{n} \min\{a_i b_j, a_j b_i\}$$

Problem 5.294 (USAMO 1999, problem 4). Let $n > 3$ be an integer and $a_1, \ldots, a_n$ be positive real numbers such that

$$a_1 + \ldots + a_n \geqslant n \quad \text{and}$$
$$a_1^2 + \ldots + a_n^2 \geqslant n^2$$

Prove that $\max\{a_1, \ldots, a_n\} \geqslant 2$.

Problem 5.295 (USAMO 1996, problem 3). Let $a_1, \ldots, a_n$ be real numbers in the interval $(0, \pi/2)$ such that

$$\tan(a_0 - \frac{\pi}{4}) + \ldots + \tan(a_1 - \frac{\pi}{4}) \geqslant n - 1$$

Prove that

$$\tan a_0 \cdots \tan a_n \geqslant n^{n+1}$$

Solution. See (3.4).

PROBLEM 5.296 (USAMO 1994, problem 4). Let $a_1, \ldots, a_n$ be a sequence of positive real numbers such that

$$\sum_{i=1}^{n} a_i^2 \geqslant \sqrt{n}$$

for all n. Prove that

$$\sum_{i=1}^{n} a_i^2 > \frac{1}{4}(1 + \frac{1}{2} + \ldots + \frac{1}{n})$$

PROBLEM 5.297 (USAMO 1993, problem 5). Let $a_0, \ldots, a_n$ be positive real numbers such that $a_{i-1}a_{i+1} \leqslant a_i^2$ (such a sequence is called *log concave*). Show that for $n > 1$,

$$\frac{a_0 + \ldots + a_n}{n+1} \frac{a_1 + \ldots + a_{n-1}}{n-1} \geqslant \frac{a_0 + \ldots + a_{n-1}}{n} \frac{a_1 + \ldots + a_n}{n}$$

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Glossary

ABMO Albania-Balkan Mathematical Olympiad 57

AlMO Albania Mathematical Olympiad 57

AMC Abel's Mathematical Contest (Norwegian Mathematical Olympiad) 57

APC Austrian-Polish Competition 57

APMO Asian Pacific Mathematical Olympiad 57

AzNO Azerbaijan National Olympiad 57

BkMO Balkan Mathematical Olympiad 57

BMO British Mathematical Olympiad 57

BNO Brazil National Olympiad 57

BrNO Belarusian National Olympiad 57

BuNO Bulgarian National Olympiad 57

ChMO China Mathematical Olympiad 57

Elementary polynomials All symmetric polynomials in $x_1, \dots, x_n$ can be written as a sum of *elementary polynomials*. Elementary polynomials E_k are defined as

$$\begin{aligned}
E_0(x_1, \dots, x_n) &= 1 \\
E_1(x_1, \dots, x_n) &= x_1 + \dots + x_n \\
E_2(x_1, \dots, x_n) &= x_1x_2 + x_1x_3 + \dots + x_{n-1}x_n \\
&= \sum_{1 \leqslant i < j \leqslant n} x_i x_j \\
&\vdots \\
E_k(x_1, \dots, x_n) &= \sum_{1 \leqslant i_1 < \dots < i_n \leqslant n} x_{i_1} \cdots x_{i_n}
\end{aligned}$$

If $k > n$, then $E_k(x_1, \dots, x_n) = 0$. A special case which is very useful for us is

$$\begin{aligned}
E_1(a, b, c) &= a + b + c \\
E_2(a, b, c) &= ab + bc + ca \\
E_3(a, b, c) &= abc
\end{aligned}$$

We can show that any symmetric polynomial can be written as a sum of elementary polynomials using induction. The main idea behind this is to show that

$$\sigma_k = x_1^k + \dots + x_n^k$$

can be represented in terms of $E_0, \dots, E_k$. For $k \in \{0, 1\}$, the proof is obvious. 48

Global maximum and minimum A *global minimum* (*absolute minimum*) and a *global maximum* (resp. *absolute maximum*) is the smallest possible value of a set or a function. 47

IMO International Mathematical Olympiad 57

IrMO Iranian Mathematical Olympiad 57

MMO Mediterranean Mathematical Olympiad 57

PMO Poland Mathematical Olympiad 57

RNO Romania National Olympiad 57

SgMO Singapore Mathematical Olympiad 57

TNO Taiwan National Olympiad 57

USAMO United States of America Mathematical Olympiad 57

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